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We continue Lecture 18 with logarithms of nonvanishing functions. We then compare logarithms with their local power series and prove that *every holomorphic function is analytic*. The same integral formula for the coefficients gives Cauchy's estimates and Liouville's theorem.

1 Logarithms of nonvanishing analytic functions

The same construction works for a nonvanishing function f , replacing $1/z$ by its *logarithmic derivative* f'/f . As in the handwritten notes, we assume here that f is *analytic*. Then f' is analytic, so f'/f is holomorphic when f has no zeros. This uses only facts about power series already established in the course.

Theorem 1.1. *Let $U \subseteq \mathbb{C}$ be simply connected, and let f be analytic on U with $f(z) \neq 0$ for all $z \in U$. Choose $z_0 \in U$ and $w_0 \in \mathbb{C}$ satisfying $e^{w_0} = f(z_0)$. Then*

$$L_f(z) = w_0 + \int_{z_0}^z \frac{f'(\zeta)}{f(\zeta)} d\zeta$$

is holomorphic on U , satisfies $L_f(z_0) = w_0$, and has the properties

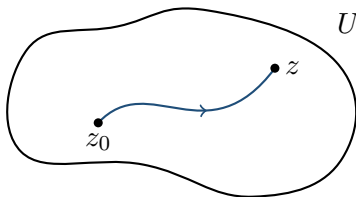
$$L'_f(z) = \frac{f'(z)}{f(z)}, \quad e^{L_f(z)} = f(z).$$

Thus L_f is a holomorphic logarithm of f on U .

Proof. The primitive theorem from Lecture 18 applies to f'/f , so the integral is path independent and gives the asserted derivative. Compute

$$\begin{aligned} \frac{d}{dz}(e^{-L_f(z)} f(z)) &= -L'_f(z) e^{-L_f(z)} f(z) + e^{-L_f(z)} f'(z) \\ &= e^{-L_f(z)} \left[-\frac{f'(z)}{f(z)} f(z) + f'(z) \right] = 0. \end{aligned}$$

Since U is connected, $e^{-L_f(z)}f(z)$ is constant. Evaluating at z_0 gives $e^{-w_0}f(z_0) = 1$. Hence $e^{L_f(z)} = f(z)$. $\square$



Integrate f'/f along any path in the simply connected domain U .

1.1 Dependence on the choices

Proposition 1.2 (Uniqueness up to $2\pi i\mathbb{Z}$). *Let U be connected and open. If L_f and $\tilde{L}_f$ are two holomorphic logarithms of the same nonvanishing function f on U , then*

$$\tilde{L}_f(z) - L_f(z) = 2\pi i k \quad (z \in U)$$

for one fixed integer k . In particular, prescribing the value at one point determines the logarithm uniquely.

Proof. Differentiating $e^{L_f} = e^{\tilde{L}_f} = f$ shows that both derivatives are f'/f . Their difference has derivative zero, so it is a constant C on U . Since $e^C = 1$, we have $C = 2\pi i k$ for some $k \in \mathbb{Z}$. $\square$

If we change the starting point from z_0 to z_1 and choose the new initial value to be $L_f(z_1)$, we recover exactly the same function:

$$L_f(z_1) + \int_{z_1}^z \frac{f'(\zeta)}{f(\zeta)} d\zeta = L_f(z).$$

Choosing another allowed initial value changes the entire logarithm by one constant multiple of $2\pi i$.

Example 1.3 (Different branches on the same slit plane). For $f(z) = z$ on $U_0 = \mathbb{C} \setminus (-\infty, 0]$, take $z_0 = 1$. The choice $w_0 = 0$ gives $\text{Log } z = \ln r + i\theta$, with $-\pi < \theta < \pi$. The choice $w_0 = 2\pi i$ gives

$$L(z) = \text{Log } z + 2\pi i = \ln r + i\theta + 2\pi i.$$

Both exponentiate to z , and both have derivative $1/z$.

Exercise 1.4. Let f be a nonvanishing analytic function on a simply connected domain U , and let $n \geq 2$ be an integer. Use a holomorphic logarithm L_f to construct a holomorphic function g on U such that $g^n = f$. How does changing L_f by $2\pi i k$ change the root g ?

2 Logarithms and products

Do logarithms satisfy the familiar rule $L_{f_1 f_2} = L_{f_1} + L_{f_2}$? The answer depends on the initial values. Let f_1, f_2 be nonvanishing analytic functions on the same simply connected domain U , and fix $z_0 \in U$. Choose

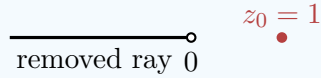
$$e^{w_1} = f_1(z_0), \quad e^{w_2} = f_2(z_0), \quad e^{w_{12}} = f_1(z_0)f_2(z_0).$$

The corresponding integral constructions give

$$\begin{aligned} L_{f_1 f_2}(z) &= w_{12} + \int_{z_0}^z \frac{(f_1 f_2)'(\zeta)}{f_1(\zeta)f_2(\zeta)} d\zeta \\ &= w_{12} + \int_{z_0}^z \left(\frac{f_1'(\zeta)}{f_1(\zeta)} + \frac{f_2'(\zeta)}{f_2(\zeta)} \right) d\zeta \\ &= L_{f_1}(z) + L_{f_2}(z) + (w_{12} - w_1 - w_2). \end{aligned}$$

Thus the product rule holds exactly if we choose $w_{12} = w_1 + w_2$. This is always an allowed choice, since $e^{w_1 + w_2} = f_1(z_0)f_2(z_0)$. For independently chosen branches, $w_{12} - w_1 - w_2$ may instead be a nonzero multiple of $2\pi i$.

Example 2.1 (The normalized logarithm on the slit plane). On $U_0 = \mathbb{C} \setminus (-\infty, 0]$, the choices $f(z) = z$, $z_0 = 1$, and $w_0 = 0$ give the principal logarithm $\text{Log } z$ of Lecture 18. Its normalization is $\text{Log } 1 = 0$.



Prescribing principal values separately does not guarantee the product rule: recall the example $z = w = e^{3\pi i/4}$ from Lecture 18.

3 The logarithm as a local power series

We know the real logarithm has a power series near 1. Start directly with its complex counterpart:

$$L_0(1 + w) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} w^n.$$

The radius of convergence is 1, because $\limsup_{n \rightarrow \infty} (1/n)^{1/n} = 1$. For $|w| < 1$, termwise differentiation and the geometric series give

$$\frac{d}{dw} L_0(1 + w) = \sum_{n=1}^{\infty} (-1)^{n-1} w^{n-1} = \frac{1}{1 + w}, \quad L_0(1) = 0.$$

3.1 Comparison with the integral construction

Let L be a holomorphic logarithm of z on a domain $U \subseteq \mathbb{C} \setminus \{0\}$, and fix $z_0 \in U$. Put $w_0 = L(z_0)$, so $e^{w_0} = z_0$. Choose $r > 0$ such that $D(z_0, r) \subseteq U$ and $r < |z_0|$. On this disc define

$$\begin{aligned} F(z) &:= w_0 + L_0\left(1 + \frac{z - z_0}{z_0}\right) \\ &= w_0 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{nz_0^n} (z - z_0)^n. \end{aligned}$$

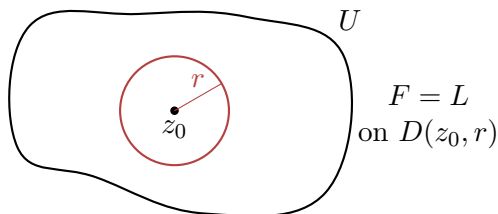
This is analytic, and

$$F(z_0) = w_0, \quad F'(z) = \frac{1}{z_0} \frac{1}{1 + (z - z_0)/z_0} = \frac{1}{z}.$$

But $L'(z) = 1/z$ as well. Therefore $F - L$ has derivative zero on the connected disc $D(z_0, r)$ and vanishes at z_0 . It follows that

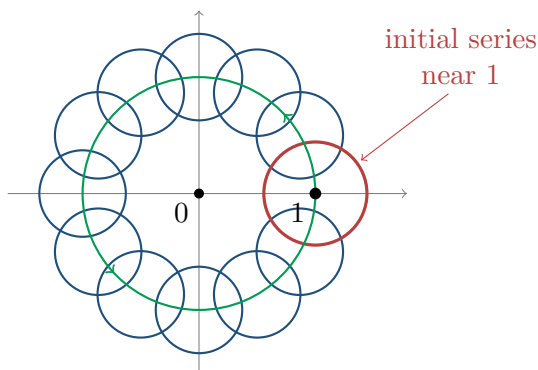
$$L(z) = L(z_0) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{nz_0^n} (z - z_0)^n \quad (|z - z_0| < r).$$

In particular, every holomorphic branch of the logarithm is analytic. This conclusion uses the explicit power series, before the general holomorphic-implies-analytic theorem has been proved.



3.2 The picture of overlapping discs

The handwritten sketch shows local logarithms on overlapping discs along a circle around 0. On each new disc we can adjust the initial value by a multiple of $2\pi i$ so that the local logarithms agree on the overlap. Their derivatives are all $1/z$, so matching at one point makes them agree on the connected overlap.



Local branches can be matched along a path around the origin.

After one counterclockwise turn, the continued value has increased by

$$\int_{|z|=1} \frac{dz}{z} = 2\pi i.$$

Thus local agreement along the path does not give a single-valued logarithm on all of $\mathbb{C} \setminus \{0\}$. This is the obstruction that simple connectedness removes in the integral construction.

4 Holomorphic functions are analytic

The logarithm illustrates a general phenomenon. Cauchy's integral formula produces a convergent power series for any holomorphic function.

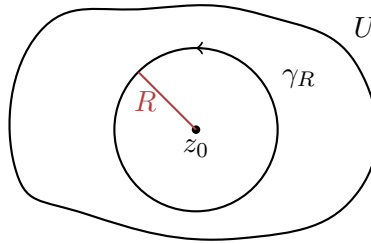
Theorem 4.1 (Power series expansion and Cauchy estimates). *Let $U \subseteq \mathbb{C}$ be open and f holomorphic on U . Suppose $\overline{D(z_0, R)} \subseteq U$, where $R > 0$, and let $\gamma_R(t) = z_0 + Re^{it}$, $0 \leq t \leq 2\pi$, be the positively oriented boundary circle. Set*

$$a_n := \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \quad (n \geq 0), \quad M_R := \max_{\zeta \in \gamma_R} |f(\zeta)|.$$

Then

1. $|a_n| \leq \frac{M_R}{R^n}$ for all $n \geq 0$ (Cauchy estimates);
2. $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ for $|z - z_0| < R$.

The series converges uniformly on each closed disc $\overline{D(z_0, s)}$ with $0 < s < R$. Consequently, f is analytic on U .



The closed-disc condition ensures that f is holomorphic on a neighborhood of the contour and its interior. For $n = 0$, the coefficient formula is simply Cauchy's formula at the center: $a_0 = f(z_0)$.

4.1 An immediate consequence: Liouville's theorem

Corollary 4.2 (Liouville's theorem). *Every bounded entire function is constant.*

Proof using Theorem 4.1. Suppose f is holomorphic on $\mathbb{C}$ and $|f(z)| \leq M$ for all z . Expand at $z_0 = 0$. For every $R > 0$, the theorem gives

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (|z| < R), \quad |a_n| \leq \frac{M}{R^n}.$$

The coefficients are independent of R : power series representing the same function near 0 have unique coefficients (equivalently, $a_n = f^{(n)}(0)/n!$). For each fixed $n \geq 1$, let $R \rightarrow \infty$. Then $a_n = 0$. Since R can be arbitrarily large, $f(z) = a_0$ on all of $\mathbb{C}$. $\square$

We now prove the expansion theorem used above.

5 Proof of the power series expansion

Proof of Theorem 4.1. Step 1: estimate the coefficients. On γ_R , we have $|\zeta - z_0| = R$, and the contour has length $2\pi R$. The estimation lemma gives

$$|a_n| \leq \frac{1}{2\pi} (2\pi R) \frac{M_R}{R^{n+1}} = \frac{M_R}{R^n}.$$

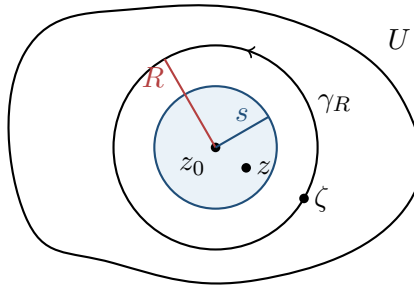
Step 2: expand the Cauchy kernel. Fix $0 < s < R$. For $|z - z_0| \leq s$ and $\zeta \in \gamma_R$,

$$\left| \frac{z - z_0}{\zeta - z_0} \right| \leq \frac{s}{R} < 1.$$

Consequently,

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} \\ &= \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}. \end{aligned}$$

The absolute value of the n th term is at most $R^{-1}(s/R)^n$. Since these bounds form a convergent geometric series, the kernel series converges uniformly for all such z and ζ .



The smaller disc gives a uniform ratio $s/R < 1$ on the contour.

Step 3: integrate term by term. Cauchy's integral formula, recalled in Lecture 18, says

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(\zeta)}{\zeta - z} d\zeta \quad (|z - z_0| < R).$$

The function f is bounded on γ_R . Multiplying the uniformly convergent kernel series by $f(\zeta)$ therefore preserves uniform convergence, allowing termwise integration:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\gamma_R} \sum_{n=0}^{\infty} \frac{f(\zeta)(z - z_0)^n}{(\zeta - z_0)^{n+1}} d\zeta \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\gamma_R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n \\ &= \sum_{n=0}^{\infty} a_n (z - z_0)^n. \end{aligned}$$

More explicitly, the integrated geometric-series remainder satisfies

$$\sup_{|z - z_0| \leq s} \left| f(z) - \sum_{n=0}^N a_n (z - z_0)^n \right| \leq M_R \frac{(s/R)^{N+1}}{1 - s/R} \rightarrow 0.$$

This also proves uniform convergence on the smaller closed disc. Every point of $D(z_0, R)$ lies in such a disc, so the expansion holds throughout $D(z_0, R)$. Finally, every $z_0 \in U$ admits some $R > 0$ with $\overline{D(z_0, R)} \subseteq U$. Thus f is analytic on U . $\square$

Remark. We already knew that analytic functions are holomorphic by termwise differentiation of power series. The theorem proves the converse. In particular, all derivatives of a holomorphic function are themselves holomorphic. We may now replace “analytic” by “holomorphic” in the logarithm theorem at the beginning of this lecture.