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We review the homotopy form of Cauchy's theorem, recall the local Cauchy integral formula, and justify the comparison of circles used in its proof. We then continue with global primitives and complex logarithms. Throughout, homotopies have *fixed endpoints*, as in Lecture 17.

1 Review: the homotopy form of Cauchy's theorem

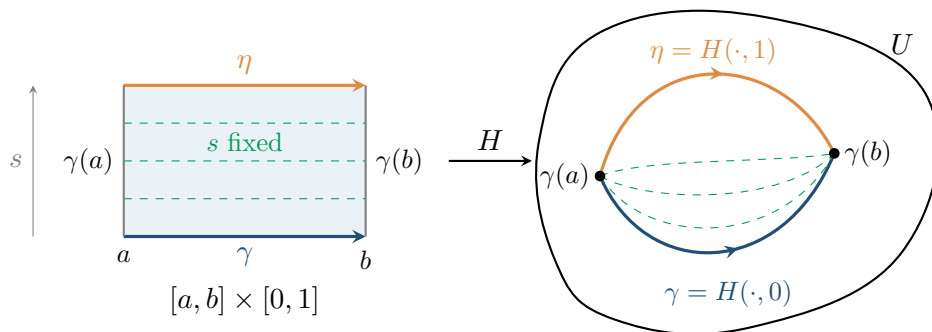
Theorem 1.1 (Homotopy form of Cauchy's theorem). *Let $U \subseteq \mathbb{C}$ be open and f holomorphic on U . If continuous curves $\gamma, \eta: [a, b] \rightarrow U$ are homotopic with fixed endpoints in U , then*

$$\int_{\gamma} f(z) dz = \int_{\eta} f(z) dz.$$

Recall that the homotopy is a continuous map $H: [a, b] \times [0, 1] \rightarrow U$ with

$$H(t, 0) = \gamma(t), \quad H(t, 1) = \eta(t), \quad H(a, s) = \gamma(a), \quad H(b, s) = \gamma(b).$$

The last two equalities hold for every $s \in [0, 1]$.



Horizontal slices map to curves joining $\gamma(a)$ to $\gamma(b)$; vertical edges collapse to these endpoints. The proof in Lecture 17 used local primitives, compactness, and uniform continuity.

Definition 1.2. A nonempty open set $U \subseteq \mathbb{C}$ is *simply connected* if it is connected and every closed continuous curve in U is *null-homotopic in U* : it can be deformed within U , with its base point fixed, to the constant curve at that base point.

Corollary 1.3. Let $U \subseteq \mathbb{C}$ be open and f holomorphic on U . If γ is a closed continuous curve that is null-homotopic in U , then $\int_{\gamma} f(z) dz = 0$. In particular, this holds for every closed continuous curve when U is simply connected.

Proof. A null-homotopic curve can be deformed, with its base point fixed, to a constant curve. The theorem preserves the integral, and a constant curve has integral zero. $\square$

2 The local Cauchy integral formula

Theorem 2.1 (Local Cauchy integral formula). Let $D = D(c, R)$, let U be an open set containing $\overline{D}$, and let f be holomorphic on U . If $z_0 \in D$ and $\gamma(t) = c + Re^{it}$, $0 \leq t \leq 2\pi$, then

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} dz.$$

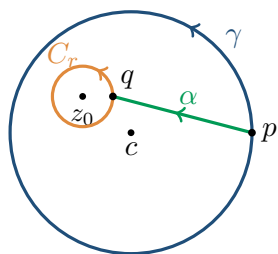
The proof in Lecture 16 hinged on a comparison of integrals over different circles. Fix $0 < r < R - |z_0 - c|$ and let $C_r(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi$. The outer circle γ has basepoint $p = c + R$ and the inner circle C_r has basepoint $q = z_0 + r$. We would like to establish that for every function h holomorphic on $U \setminus \{z_0\}$,

$$\int_{\gamma} h(z) dz = \int_{C_r} h(z) dz. \quad (2.1)$$

The issue is that the two circles have different base points: $p \neq q$, since $|z_0 - c| + r < R$. Thus they cannot be homotopic with fixed endpoints in our sense.

To fix this issue, we consider the straight segment α from p to q :

$$\alpha(s) = (1 - s)p + sq, \quad 0 \leq s \leq 1.$$



Compare the outer circle with a small circle around z_0 , always avoiding z_0 .
 α : the connecting path from $p = c + R$ to $q = z_0 + r$.

Write $\gamma * \alpha$ for the path that follows γ and then α , and similarly for $\alpha * C_r$, assigning each piece half of $[0, 1]$. Both paths run from p to q . We now construct a homotopy between them in $U \setminus \{z_0\}$ with fixed endpoints.

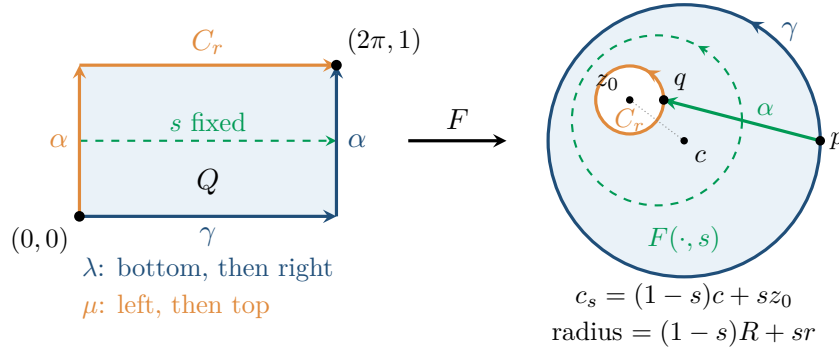
Consider the continuous map

$$F(t, s) = (1 - s)c + sz_0 + ((1 - s)R + sr)e^{it}, \quad 0 \leq t \leq 2\pi, \quad 0 \leq s \leq 1.$$

Put $d = |z_0 - c| < R$. The estimates

$$\begin{aligned} |F(t, s) - c| &\leq sd + (1 - s)R + sr \leq R, \\ |F(t, s) - z_0| &\geq (1 - s)(R - d) + sr > 0 \end{aligned}$$

show that F takes values in $\overline{D} \setminus \{z_0\}$, hence in $U \setminus \{z_0\}$. Its bottom and top edges give γ and C_r , and both vertical edges give the connecting path: $F(0, s) = F(2\pi, s) = \alpha(s)$.



Let $Q = [0, 2\pi] \times [0, 1]$, and write $\lambda, \mu: [0, 1] \rightarrow Q$ for the two boundary paths shown above, with each edge traversed linearly in half the parameter interval. The path λ follows the bottom and right edges; μ follows the left and top edges. The edge labels in the rectangle indicate the images of the edges under F , so

$$F \circ \lambda = \gamma * \alpha, \quad F \circ \mu = \alpha * C_r.$$

Both paths start at $(0, 0)$ and end at $(2\pi, 1)$. Since Q is convex, λ and μ are homotopic in Q with fixed endpoints, e.g. by the straight-line homotopy

$$H(t, s) = (1 - s)\lambda(t) + s\mu(t), \quad 0 \leq t, s \leq 1.$$

Composing H with F gives a continuous map $[0, 1] \times [0, 1] \xrightarrow{H} Q \xrightarrow{F} U \setminus \{z_0\}$, namely

$$C(t, s) = F((1 - s)\lambda(t) + s\mu(t)), \quad 0 \leq t, s \leq 1.$$

This is a homotopy between $F \circ \lambda = \gamma * \alpha$ and $F \circ \mu = \alpha * C_r$ in $U \setminus \{z_0\}$, with $C(0, s) = p$ and $C(1, s) = q$ for every s . Thus Theorem 1.1, applied on the open set $U \setminus \{z_0\}$, and additivity of integrals give

$$\int_{\gamma} h(z) dz + \int_{\alpha} h(z) dz = \int_{\alpha} h(z) dz + \int_{C_r} h(z) dz \quad (h \in \mathcal{H}ol(U \setminus \{z_0\})).$$

Cancelling the connecting-path integrals yields (2.1):

$$\int_{\gamma} h(z) dz = \int_{C_r} h(z) dz \quad \text{for every } h \in \mathcal{H}ol(U \setminus \{z_0\}).$$

Applying this to $h(z) = (f(z) - f(z_0))/(z - z_0)$ and to $h(z) = 1/(z - z_0)$ justifies both circle comparisons in the proof from Lecture 16, and so completes the proof of Theorem 2.1.

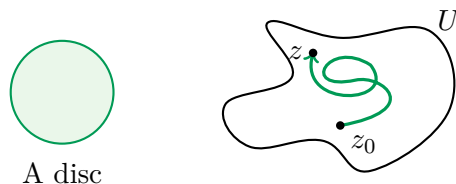
3 Primitives on simply connected domains

We know that a holomorphic function has a primitive on each disc in its domain, by the Cauchy–Goursat theorem for rectangles in discs. The homotopy theorem now lets us replace the disc by any simply connected domain.

Theorem 3.1. *Let $U \subseteq \mathbb{C}$ be simply connected and let f be holomorphic on U . Then f has a primitive on all of U . More precisely, fix $z_0 \in U$ and define*

$$g(z) = \int_{z_0}^z f(\zeta) d\zeta,$$

where the integral is taken along any continuous curve in U from z_0 to z . This is well-defined, and $g' = f$ on U .



Proof. Existence of paths. We claim that a connected open subset of $\mathbb{C}$ is polygonally connected.¹

Here is a proof using the standard definition of connectedness below.

Remark (Connected open sets are polygonally connected). For Theorem 3.1, we only need that connected open subsets of $\mathbb{C}$ are *path connected*. The proof below gives the stronger statement that they are *polygonally connected*.

Call a set $S \subseteq \mathbb{C}$ *connected* if it cannot be split into two disjoint, nonempty, relatively open pieces. That is, whenever $S = X \cup Y$ with $X \cap Y = \emptyset$, $X = V \cap S$ and $Y = W \cap S$ for some open sets $V, W \subseteq \mathbb{C}$, then $X = \emptyset$ or $Y = \emptyset$.

Let A be the set of points of U that can be reached from z_0 by a polygonal path in U , and let $B = U \setminus A$.

- *A is open.* If $a \in A$, choose a disc $D(a, \varepsilon) \subseteq U$. Each w in this disc is reached by following a polygonal path from z_0 to a and then the segment $[a, w]$, which lies in the disc.
- *B is open.* If $b \in B$, choose a disc $D(b, \varepsilon) \subseteq U$. If some w in this disc were in A , then appending the segment $[w, b]$ would reach b , which is impossible. So the whole disc lies in B .

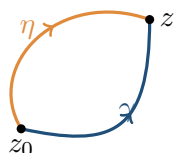
Thus $U = A \cup B$ is a union of disjoint open sets, and $A \neq \emptyset$ since $z_0 \in A$. As U is connected, $B = \emptyset$, i.e. $A = U$. Finally, any two points $z, w \in U$ are joined by following a path from z back to z_0 and then one from z_0 to w .

¹An open set $U \subseteq \mathbb{C}$ is *polygonally connected* if any two points $z, w \in U$ can be joined by a *polygonal path* in U : a finite chain of line segments $[w_0, w_1], [w_1, w_2], \dots, [w_{n-1}, w_n]$, each contained in U , with $w_0 = z$ and $w_n = w$. A polygonal path is piecewise C^1 , so in particular it is a continuous curve along which we can integrate.

Independence of the path. Suppose γ and η both run from z_0 to z in U . Write η^{-1} for η traversed backwards. Then $\gamma * \eta^{-1}$ is a closed curve in U . It is null-homotopic because U is simply connected, so Corollary 1.3 gives

$$0 = \int_{\gamma * \eta^{-1}} f(\zeta) d\zeta = \int_{\gamma} f(\zeta) d\zeta - \int_{\eta} f(\zeta) d\zeta.$$

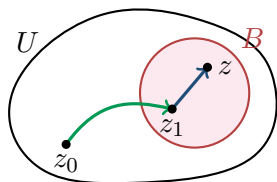
Thus the definition of $g(z)$ does not depend on the chosen curve.



Follow γ from z_0 to z ,
then η^{-1} back to z_0 .

Differentiating locally. Fix $z_1 \in U$ and choose a disc $B = D(z_1, \varepsilon) \subseteq U$. Let G be a primitive of f on B . For $z \in B$, use a fixed path from z_0 to z_1 followed by the straight segment from z_1 to z . Path independence gives

$$g(z) = g(z_1) + \int_{[z_1, z]} f(\zeta) d\zeta = g(z_1) + G(z) - G(z_1).$$



A fixed path to z_1 ,
then a short segment inside B .

The two terms involving z_1 are constant as z varies in B . Consequently $g'(z) = G'(z) = f(z)$ on B . Since z_1 was arbitrary, g is a primitive on all of U . $\square$

4 Complex logarithms

Recall that

$$e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}, \quad e^{x+iy} = e^x(\cos y + i \sin y).$$

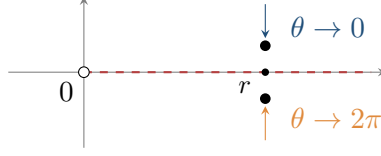
The exponential is never zero and has period $2\pi i$. We would like to invert it. If $z = re^{i\theta} \neq 0$, then all solutions of $e^w = z$ are

$$w = \ln r + i(\theta + 2\pi k), \quad k \in \mathbb{Z}.$$

Here $\ln$ denotes the ordinary real logarithm. The difficulty is to choose the argument continuously as z varies.

4.1 Why a branch is needed

If we always choose $\theta \in [0, 2\pi)$, the function $\ln|z| + i\theta$ jumps across the positive real axis. At a positive real number r , the limits from above and below are respectively $\ln r$ and $\ln r + 2\pi i$.



In fact, no continuous function L on $\mathbb{C} \setminus \{0\}$ satisfies $e^{L(z)} = z$ for all $z \neq 0$. If such an L existed, then

$$L(e^{it}) - it \in 2\pi i\mathbb{Z} \quad (0 \leq t \leq 2\pi).$$

This continuous function of t takes values in a discrete set, so it is constant. But its values at 0 and 2π differ by $-2\pi i$, since $e^0 = e^{2\pi i} = 1$. This is a contradiction.

Definition 4.1. A *holomorphic branch of the logarithm* on an open set $U \subseteq \mathbb{C} \setminus \{0\}$ is a holomorphic function $L: U \rightarrow \mathbb{C}$ satisfying $e^{L(z)} = z$ for every $z \in U$.

Differentiating this identity gives $e^{L(z)} L'(z) = 1$, hence $L'(z) = 1/z$. This suggests constructing a logarithm by integrating $1/z$.

4.2 Constructing a logarithm on a simply connected domain

Theorem 4.2. Let $U \subseteq \mathbb{C} \setminus \{0\}$ be simply connected. Choose $z_0 \in U$ and $w_0 \in \mathbb{C}$ with $e^{w_0} = z_0$. Then

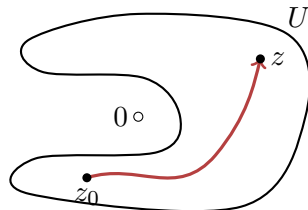
$$L(z) = w_0 + \int_{z_0}^z \frac{d\zeta}{\zeta}$$

defines a holomorphic branch of the logarithm on U , with $L(z_0) = w_0$ and $L'(z) = 1/z$.

Proof. The function $1/z$ is holomorphic on U , so Theorem 3.1 shows that the integral is path independent and $L' = 1/z$. To verify the exponential identity, differentiate:

$$\frac{d}{dz}(ze^{-L(z)}) = e^{-L(z)} - ze^{-L(z)}L'(z) = 0.$$

Since U is connected, $ze^{-L(z)}$ is constant. At z_0 , its value is $z_0e^{-w_0} = 1$, so $e^{L(z)} = z$ on U . □



$0 \notin U$; all paths stay in U .
The choice $L(z_0) = w_0$ fixes the branch.

4.3 The principal branch

Take the slit plane

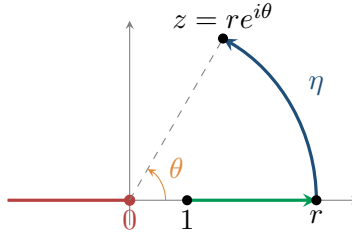
$$U_0 = \mathbb{C} \setminus (-\infty, 0], \quad z_0 = 1, \quad w_0 = 0.$$

The entire nonpositive real axis, including 0, is removed. On this domain choose the unique argument $\text{Arg } z \in (-\pi, \pi)$, which varies continuously. The coordinates $z \mapsto (\ln |z|, \text{Arg } z)$ identify U_0 with the convex strip $\mathbb{R} \times (-\pi, \pi)$; the continuous inverse is $(x, \theta) \mapsto e^x e^{i\theta}$. Straight-line contractions in the strip, transported back to U_0 , contract any loop while fixing its base point. Thus U_0 is simply connected.

By Theorem 4.2, the *principal logarithm* is

$$\text{Log } z = \int_1^z \frac{d\zeta}{\zeta}, \quad z \in U_0.$$

Write $z = re^{i\theta}$ with $r > 0$ and $-\pi < \theta < \pi$. Use a path consisting of the positive real segment from 1 to r , followed by the circular arc from r to z .



On the real segment, the integral is $\int_1^r dx/x = \ln r$; this also holds when $0 < r < 1$, with the segment traversed towards r . Parametrize the arc by $\eta(u) = re^{iu\theta}$, $0 \leq u \leq 1$. This parametrization works for either sign of θ , and

$$\int_{\eta} \frac{d\zeta}{\zeta} = \int_0^1 \frac{i\theta r e^{iu\theta}}{r e^{iu\theta}} du = i\theta.$$

Therefore

$$\boxed{\text{Log } z = \ln |z| + i \text{Arg } z, \quad -\pi < \text{Arg } z < \pi.}$$

The integral construction proves that this branch is holomorphic; the polar formula evaluates it explicitly.

Warning. The identity $\text{Log}(zw) = \text{Log } z + \text{Log } w$ need not hold even when $z, w, zw \in U_0$. For example, $z = w = e^{3\pi i/4}$ and $zw = -i$ all lie in U_0 , but

$$\text{Log}(zw) = -\frac{\pi i}{2}, \quad \text{Log } z + \text{Log } w = \frac{3\pi i}{2}.$$

The two sides differ by $2\pi i$.