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We begin with the definition of homotopy from Lecture 16, then prove that integrals of holomorphic functions are unchanged by homotopy. Throughout, $U \subseteq \mathbb{C}$ is open and $a < b$. Integrals along continuous curves have the meaning defined in Lecture 16, using local primitives on discs.

1 Homotopy of curves

1.1 Homotopy with fixed endpoints

Definition 1.1. Let $\gamma, \eta: [a, b] \rightarrow U$ be continuous curves with $\gamma(a) = \eta(a)$ and $\gamma(b) = \eta(b)$. A *homotopy with fixed endpoints* from γ to η in U is a continuous map

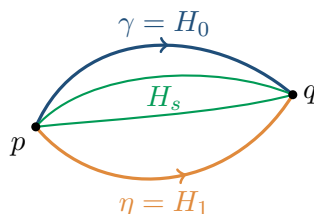
$$H: [a, b] \times [0, 1] \longrightarrow U$$

such that

$$\begin{aligned} H(t, 0) &= \gamma(t), & H(t, 1) &= \eta(t) & (a \leq t \leq b), \\ H(a, s) &= \gamma(a) = \eta(a), & H(b, s) &= \gamma(b) = \eta(b) & (0 \leq s \leq 1). \end{aligned}$$

We write $\gamma \sim \eta$ in U when such a homotopy exists. Throughout this lecture, *homotopic* always means homotopic with fixed endpoints.

For fixed s , write $H_s(t) = H(t, s)$. Thus $H_0 = \gamma$, $H_1 = \eta$, and every intermediate curve has the same endpoints. The parameter t moves along a curve; the parameter s deforms the curve. The two endpoint conditions hold for every s , not just at $s = 0$ and $s = 1$.



Example 1.2 (Straight-line homotopy). Recall that U is convex if it contains the line segment joining any two of its points. If U is convex and γ, η have the same endpoints, then

$$H(t, s) = (1 - s)\gamma(t) + s\eta(t)$$

is a homotopy with fixed endpoints in U . Indeed, it is continuous, convexity keeps its values in U , and the boundary conditions hold.

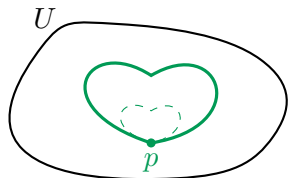
1.2 Closed curves and simple connectivity

Definition 1.3. A closed continuous curve $\gamma: [a, b] \rightarrow U$, with $p = \gamma(a) = \gamma(b)$, is *null-homotopic in U* if it is homotopic with fixed endpoints to the constant curve $c_p(t) = p$. Such a homotopy H (a *contraction*) satisfies

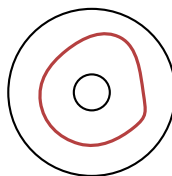
$$H(t, 0) = \gamma(t), \quad H(t, 1) = p, \quad H(a, s) = H(b, s) = p.$$

A nonempty open set $U \subseteq \mathbb{C}$ is *simply connected* if it is **connected** and every closed continuous curve in U is null-homotopic in U .

For a nonempty convex open set, put $p = \gamma(a) = \gamma(b)$. The homotopy $H(t, s) = (1 - s)\gamma(t) + sp$ contracts γ to c_p while keeping its endpoints fixed at p . A convex set is connected, so every nonempty convex open set is simply connected. Connectedness is a separate requirement: the union of two disjoint open discs has every closed curve null-homotopic, but is not simply connected. The ambient set matters: a deformation must stay in U at every stage.



A loop contracts inside U , keeping its base point p fixed.



A hole may obstruct contraction.

The right-hand picture shows a disc with a hole and suggests an obstruction; the example below proves this for the punctured disc, where the hole is a single point, by integrating $1/z$.

2 The homotopy form of Cauchy's theorem

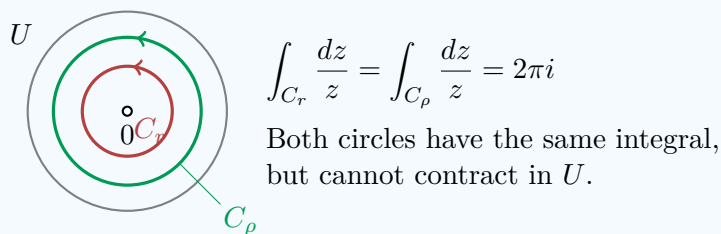
Theorem 2.1 (Homotopy form of Cauchy's theorem). *Let f be holomorphic on an open set $U \subseteq \mathbb{C}$, and let $\gamma, \eta: [a, b] \rightarrow U$ be continuous curves. If γ and η are homotopic with fixed endpoints in U , then*

$$\boxed{\int_{\gamma} f(z) dz = \int_{\eta} f(z) dz.} \quad (2.1)$$

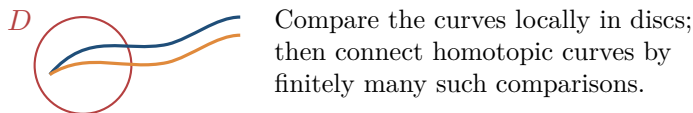
Example 2.2 (A puncture prevents contraction). Let $U = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and $0 < r < 1$. For the circle $C_r(t) = re^{it}$, $0 \leq t \leq 2\pi$,

$$\int_{C_r} \frac{dz}{z} = \int_0^{2\pi} \frac{ire^{it}}{re^{it}} dt = 2\pi i.$$

Since $1/z$ is holomorphic on U , Theorem 2.1 shows that C_r is not null-homotopic in U . Indeed, a contraction would force this integral to equal the integral over a constant curve, which is zero. Thus U is not simply connected. For any $0 < \rho < 1$, the same calculation gives $\int_{C_\rho} dz/z = 2\pi i$. The circles have different base points when $r \neq \rho$, so they cannot be compared directly by a fixed-endpoint homotopy. In Lecture 18 we will compare their integrals by adding a connecting path before applying the fixed-endpoint homotopy theorem. Null-homotopy is sufficient for the integral to vanish, but not necessary: for example, $\int_{C_r} 1 dz = 0$, although C_r is not null-homotopic in U .



Proof strategy. We first compare curves that are close in the sense that corresponding pieces lie in the same discs. We then use compactness and uniform continuity to subdivide a homotopy into finitely many such comparisons. No differentiability of the homotopy or its intermediate curves is required.



2.1 Step 1: curves in a common chain of discs

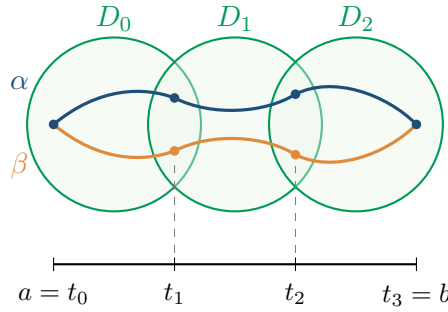
Definition 2.3 (Close curves). Two continuous curves $\alpha, \beta: [a, b] \rightarrow U$ with the same endpoints are called *close* if there exist a subdivision

$$a = t_0 < t_1 < \cdots < t_n = b$$

and open discs $D_0, \dots, D_{n-1} \subseteq U$ such that, for every j ,

$$\alpha([t_j, t_{j+1}]) \cup \beta([t_j, t_{j+1}]) \subseteq D_j.$$

Thus one subdivision and one chain of discs serve for both curves.



One subdivision; both subcurves over each interval lie in the same disc.

Lemma 2.4. If α and β are close and f is holomorphic on U , then $\int_{\alpha} f(z) dz = \int_{\beta} f(z) dz$.

Proof. Choose a primitive g_j of f on each D_j , using Cauchy's theorem on a disc. By the definition of integration along continuous curves,

$$\begin{aligned} \int_{\alpha} f(z) dz &= \sum_{j=0}^{n-1} [g_j(\alpha(t_{j+1})) - g_j(\alpha(t_j))], \\ \int_{\beta} f(z) dz &= \sum_{j=0}^{n-1} [g_j(\beta(t_{j+1})) - g_j(\beta(t_j))]. \end{aligned}$$

For $t \in [t_j, t_{j+1}]$, put

$$A_j(t) = g_j(\alpha(t)) - g_j(\beta(t)).$$

Subtracting the two integral formulas gives

$$\int_{\alpha} f(z) dz - \int_{\beta} f(z) dz = \sum_{j=0}^{n-1} [A_j(t_{j+1}) - A_j(t_j)]. \quad (2.2)$$

Why do the interior terms cancel? For $1 \leq j \leq n-1$, both $\alpha(t_j)$ and $\beta(t_j)$ lie in $D_{j-1} \cap D_j$. This intersection is nonempty and convex, hence connected. Since

$$(g_{j-1} - g_j)' = f - f = 0,$$

we have $g_{j-1} - g_j = C_j$ on this intersection. Consequently,

$$\begin{aligned} A_{j-1}(t_j) &= g_{j-1}(\alpha(t_j)) - g_{j-1}(\beta(t_j)) \\ &= [g_j(\alpha(t_j)) + C_j] - [g_j(\beta(t_j)) + C_j] \\ &= A_j(t_j). \end{aligned}$$

Thus the right-hand side of (2.2) equals

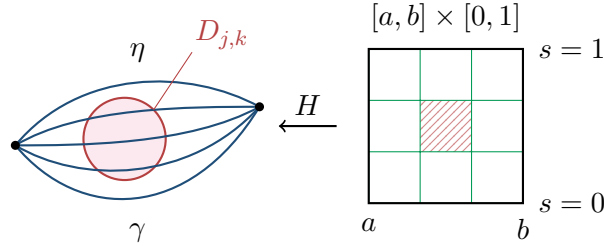
$$A_{n-1}(b) - A_0(a) + \sum_{j=1}^{n-1} \underbrace{[A_{j-1}(t_j) - A_j(t_j)]}_0.$$

The remaining endpoint terms vanish because $\alpha(a) = \beta(a)$ and $\alpha(b) = \beta(b)$. This proves the lemma. $\square$

Remark. The primitives g_{j-1} and g_j need not agree on an overlap. Their difference is constant there, which is precisely what makes the *differences of endpoint values* agree and the sum telescope.

2.2 Step 2: subdividing the homotopy

We have proved the local comparison. To compare homotopic curves, we now seek finitely many levels $0 = s_0 < \dots < s_m = 1$ such that $H(\cdot, s_k)$ and $H(\cdot, s_{k+1})$ are close for every k . The picture suggests subdividing the parameter rectangle in both directions.



Lemma 2.5 (A grid of small rectangles). *Let $H: [a, b] \times [0, 1] \rightarrow U$ be continuous. There exist subdivisions*

$$a = t_0 < \dots < t_n = b, \quad 0 = s_0 < \dots < s_m = 1$$

and open discs $D_{j,k} \subseteq U$ such that, writing $R_{j,k} = [t_j, t_{j+1}] \times [s_k, s_{k+1}]$,

$$H(R_{j,k}) \subseteq D_{j,k} \quad (0 \leq j < n, \ 0 \leq k < m).$$

Proof. **First choose a radius in the image.** The image $K = H([a, b] \times [0, 1])$ is compact. By the uniform-neighbourhood lemma from Lecture 15, choose $r > 0$ such that

$$D(w, r) \subseteq U \quad \text{for every } w \in K.$$

Then choose a mesh in the parameter rectangle. Since H is continuous on a compact rectangle, it is uniformly continuous. Choose $\delta > 0$ such that

$$\|(t, s) - (t', s')\| < \delta \implies |H(t, s) - H(t', s')| < r.$$

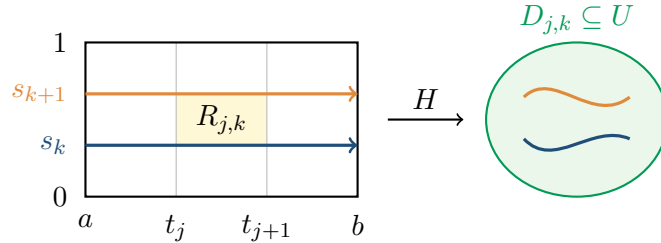
Choose both subdivisions with all their interval lengths less than $\delta/2$. Each small rectangle then has diameter less than $\sqrt{2}\delta/2 < \delta$. Set

$$D_{j,k} = D(H(t_j, s_k), r).$$

Every point (t, s) of the corresponding rectangle satisfies

$$|H(t, s) - H(t_j, s_k)| < r,$$

so its image lies in $D_{j,k} \subseteq U$, as required. The radius r and the number δ work for the entire homotopy, so one finite grid suffices. $\square$



The image of a small rectangle need not itself be a rectangle, or even have a simple shape. What matters is that it lies in one disc. In particular, the two horizontal edges map to curves in the same disc.

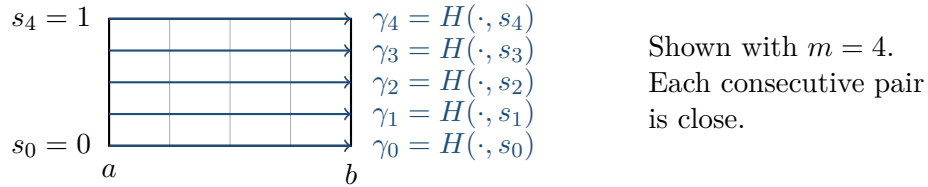
Proof of Theorem 2.1. Let H be the given homotopy and choose the grid from Lemma 2.5. For $0 \leq k \leq m$, define

$$\gamma_k(t) = H(t, s_k), \quad a \leq t \leq b.$$

Then $\gamma_0 = \gamma$ and $\gamma_m = \eta$. All these curves have the same endpoints. For $0 \leq j < n$ and $0 \leq k < m$,

$$\gamma_k([t_j, t_{j+1}]) \cup \gamma_{k+1}([t_j, t_{j+1}]) \subseteq D_{j,k}.$$

Thus consecutive curves γ_k and γ_{k+1} are close.



Lemma 2.4 gives

$$\int_{\gamma_0} f(z) dz = \int_{\gamma_1} f(z) dz = \cdots = \int_{\gamma_m} f(z) dz,$$

which is the desired equality. $\square$

3 Consequences

3.1 Null-homotopic curves

Corollary 3.1 (Cauchy's theorem for a null-homotopic curve). *If f is holomorphic on U and γ is a closed continuous curve that is null-homotopic in U , then*

$$\int_{\gamma} f(z) dz = 0.$$

In particular, if U is simply connected, the integral of every holomorphic function on U around every closed continuous curve in U is zero.

Proof. Put $p = \gamma(a) = \gamma(b)$. By definition, γ is homotopic with fixed endpoints to c_p . By Theorem 2.1, $\int_{\gamma} f(z) dz = \int_{c_p} f(z) dz = 0$. $\square$

Exercise 3.2. Let U be convex and open, and let f be holomorphic on U . Use straight-line homotopy and Theorem 2.1 to show that $\int_{\gamma} f(z) dz$ depends only on the endpoints of γ . Explain why this also proves that the integral is zero on every closed continuous curve in U .