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In Lecture 15 we established the chain-of-discs lemma for continuous curves. We now use it to define integrals of holomorphic functions along continuous curves using local primitives, then preview the local Cauchy integral formula and introduce homotopy of curves.

1 Integrating holomorphic functions along continuous curves

Definition (Partition). Let $\gamma: [a, b] \rightarrow U$ be continuous, with U open and $a < b$. Write

$$P := \{a_0, \dots, a_n\}, \quad a = a_0 < \dots < a_n = b,$$

for a *subdivision of $[a, b]$* . Using the chain-of-discs lemma from Lecture 15, we call the data

$$\mathcal{D} := (P, D_0, \dots, D_{n-1})$$

a *partition of γ* if each D_j is an open disc contained in U with $\gamma([a_j, a_{j+1}]) \subseteq D_j$. We call P the *underlying subdivision of $\mathcal{D}$* .

We recall Lemma 2.4 from Lecture 15, which guarantees that partitions exist.

Lemma 1.1 (A chain of discs along a curve). *Let $\gamma: [a, b] \rightarrow U$ be continuous, with U open and $a < b$. There exist a subdivision*

$$a = a_0 < a_1 < \dots < a_n = b$$

and open discs $D_0, \dots, D_{n-1} \subseteq U$ such that $\gamma([a_j, a_{j+1}]) \subseteq D_j$ for every j . Equivalently, every such curve admits a partition.

Proof. The image $K = \gamma([a, b])$ is compact. By the uniform-neighbourhood lemma from Lecture 15, choose $r > 0$ such that $D(z, r) \subseteq U$ for every $z \in K$. Also, γ is uniformly continuous on $[a, b]$. Thus there is $\delta > 0$ such that

$$s, t \in [a, b], \quad |s - t| < \delta \implies |\gamma(s) - \gamma(t)| < r/2.$$

Choose a subdivision with $a_{j+1} - a_j < \delta$, and set $D_j = D(\gamma(a_j), r)$. For $t \in [a_j, a_{j+1}]$, $|\gamma(t) - \gamma(a_j)| < r/2$, so the whole subcurve lies in D_j . $\square$

1.1 Definition by local primitives

Definition 1.2. Let f be holomorphic on an open set U , and let $\gamma: [a, b] \rightarrow U$ be continuous, with $a \leq b$. If $a < b$, choose a partition $\mathcal{D}$ as above. On each D_j , choose a primitive g_j of f , which exists by Cauchy's theorem on a disc. Define

$$\int_{\gamma} f(z) dz = \sum_{j=0}^{n-1} [g_j(\gamma(a_{j+1})) - g_j(\gamma(a_j))] . \quad (1.1)$$

If $a = b$, define the integral to be zero.

Remark. This definition uses the continuity of the curve and the holomorphicity of f . No derivative of γ is needed.

Proposition 1.3. *The value in (1.1) is independent of the partition $\mathcal{D}$ and the chosen local primitives. For a piecewise continuously differentiable path, it agrees with the usual contour integral.*

Proof. If $a = b$, our definition and the usual contour integral both give zero. We therefore assume $a < b$.

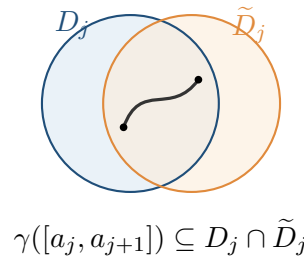
Choice of primitive. If g and h are primitives of f on one disc D , then $(g - h)' = 0$. Since D is connected, $g - h$ is constant. Consequently, for any $w_0, w_1 \in D$,

$$g(w_1) - g(w_0) = h(w_1) - h(w_0).$$

Possible changes of partition. Recall that $\mathcal{D}$ consists of subdivision points *and* one disc for each subinterval. There are two basic changes to consider:

1. *Insert a subdivision point.* Replace $[a_j, a_{j+1}]$ by $[a_j, c]$ and $[c, a_{j+1}]$, where $a_j < c < a_{j+1}$. We may use the same disc D_j for both pieces, since both subcurves still lie in D_j . Repeating this operation is called a *refinement of the subdivision*.
2. *Change a disc.* Keep the subdivision points fixed and replace D_j by another open disc $\tilde{D}_j \subseteq U$ containing the entire subcurve $\gamma([a_j, a_{j+1}])$. The discs need not contain one another; it is enough that both contain this subcurve.

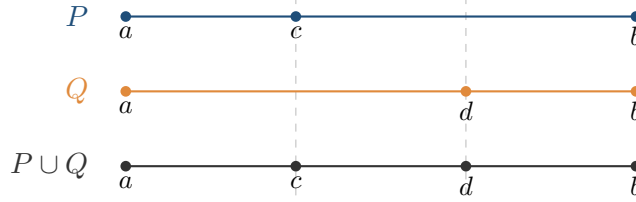
One may also remove or move subdivision points, provided the resulting data still form a partition of γ . For example, removing a point requires a disc in U containing the whole merged subcurve. We will compare such changes by refining both subdivisions to a common one.



What is a common refinement? Given two subdivisions $P = \{a_0, \dots, a_n\}$ and $Q = \{b_0, \dots, b_m\}$ of $[a, b]$, list all their points in increasing order, counting a point only once if it occurs in both:

$$P \cup Q = \{s_0, \dots, s_N\}, \quad a = s_0 < \dots < s_N = b.$$

This is their *common refinement*: it is obtained from either subdivision by inserting points. Each new subinterval $[s_k, s_{k+1}]$ lies in one subinterval of P and one of Q .



The common refinement aligns the subdivision points; it does not yet make the disc choices identical. Refine each partition by retaining its own original disc on every smaller piece. Thus each new subcurve lies in both an inherited disc D_j from the first partition and an inherited disc $\tilde{D}_\ell$ from the second:

$$\gamma([s_k, s_{k+1}]) \subseteq D_j \cap \tilde{D}_\ell.$$

We can therefore compare the two primitive increments on this intersection.

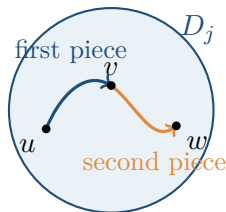
Refinement of the partition. We now check that the changes just described preserve the defining sum. Insert $c \in (a_j, a_{j+1})$, keeping the disc D_j and its primitive g_j on both new pieces. Write

$$u = \gamma(a_j), \quad v = \gamma(c), \quad w = \gamma(a_{j+1}).$$

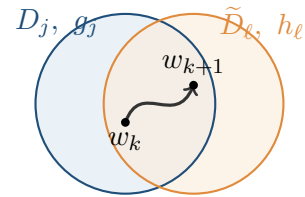
The old contribution is $g_j(w) - g_j(u)$, whereas the two new contributions add to

$$[g_j(v) - g_j(u)] + [g_j(w) - g_j(v)] = g_j(w) - g_j(u).$$

The two values at the inserted point cancel. All other terms in the sum are unchanged, so inserting one point preserves the sum. By repeating this argument, any finite refinement preserves it. If we choose different primitives on the new pieces, the first part of the proof shows that their endpoint differences are the same.



One disc and one primitive: the value at v cancels.



$g_j = h_\ell + C_k$ on the overlap

Two discs: the same constant occurs at both endpoints and cancels.

Choice of discs and partition. Consider any two partitions $\mathcal{D}$ and $\tilde{\mathcal{D}}$, with subdivisions P and Q and primitives g_j on D_j and h_ℓ on $\tilde{D}_\ell$. Refine both to $P \cup Q = \{s_0, \dots, s_N\}$, retaining their

respective discs and primitives on each smaller piece. The preceding argument shows that neither sum changes.

Fix one new subinterval $[s_k, s_{k+1}]$, as in the right-hand diagram above. It lies in an original subinterval from each partition, so its image lies in $D_j \cap \tilde{D}_\ell$ for suitable j and ℓ . This intersection is nonempty, open, and convex, hence connected. On it,

$$(g_j - h_\ell)' = f - f = 0, \quad \text{so} \quad g_j - h_\ell = C_k$$

for a constant C_k . Set $w_k = \gamma(s_k)$ and $w_{k+1} = \gamma(s_{k+1})$. Both endpoints lie in the intersection, and therefore

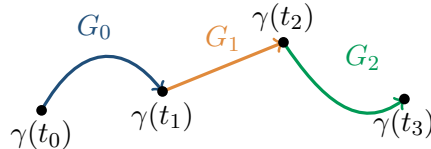
$$\begin{aligned} g_j(w_{k+1}) - g_j(w_k) &= [h_\ell(w_{k+1}) + C_k] - [h_\ell(w_k) + C_k] \\ &= h_\ell(w_{k+1}) - h_\ell(w_k). \end{aligned}$$

Thus the two refined sums agree term by term. Since refinement preserved each original sum, the original sums agree as well. This proves independence of the partition, including its disc choices.

Agreement with the usual integral. Suppose now that γ is piecewise continuously differentiable. Start with any partition used in our definition and insert all the breakpoints of γ . The defining sum is unchanged. Denote the refined subdivision by $a = t_0 < \cdots < t_M = b$, and let G_k be the inherited primitive on a disc containing $\gamma([t_k, t_{k+1}])$.

On each of these pieces, the usual fundamental theorem for a primitive applies:

$$\int_{\gamma|_{[t_k, t_{k+1}]}} f(z) dz = G_k(\gamma(t_{k+1})) - G_k(\gamma(t_k)).$$



Split at the corners and apply endpoint evaluation on each piece.

Adding these identities, additivity of the usual contour integral shows that it equals our defining sum. $\square$

Corollary 1.4 (Endpoint evaluation). *If f has a primitive g on U , then for every continuous curve $\gamma: [a, b] \rightarrow U$,*

$$\int_{\gamma} f(z) dz = g(\gamma(b)) - g(\gamma(a)).$$

In particular, the integral is zero on every closed continuous curve in U . Thus Cauchy's theorem on a disc also holds for continuous curves.

Proof. Use the restrictions of g as the local primitives. The sum telescopes. $\square$

The integral is linear in f , additive when curves are concatenated, and changes sign when the orientation is reversed. Each property follows directly from the endpoint differences in (1.1).

2 The local Cauchy integral formula: a preview

Why do we need integration over continuous curves? One reason is that we want to deform contours continuously. Here is an important application; the contour-deformation step will be justified by the homotopy theorem in the next lecture.

Theorem 2.1 (Local Cauchy integral formula). *Let $D = D(c, R)$ be an open disc, let U be an open set containing $\overline{D}$, and let f be holomorphic on U . If $z_0 \in D$ and γ is the counterclockwise boundary circle of D , then*

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} dz. \quad (2.1)$$

Proof sketch, assuming contour deformation. For $z \neq z_0$, set

$$q(z) = \frac{f(z) - f(z_0)}{z - z_0}.$$

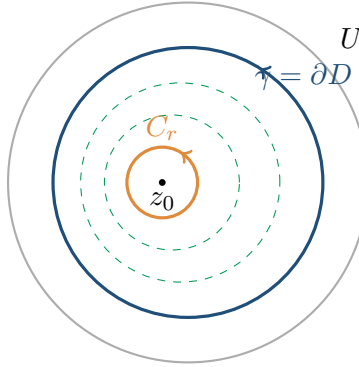
This is holomorphic on $U \setminus \{z_0\}$. The existence of $f'(z_0)$ gives

$$\lim_{z \rightarrow z_0} q(z) = f'(z_0),$$

so q is bounded near z_0 . We only need this bound; we do not yet need to prove that the extension of q at z_0 is holomorphic.

Choose $0 < r < R - |z_0 - c|$, and let $C_r(\theta) = z_0 + re^{i\theta}$, $0 \leq \theta \leq 2\pi$, oriented counterclockwise. The boundary circle γ can be continuously deformed to C_r inside $U \setminus \{z_0\}$. The contour-deformation theorem, to be proved later, gives

$$\int_{\gamma} q(z) dz = \int_{C_r} q(z) dz. \quad (2.2)$$



Take M such that $|q(z)| \leq M$ near z_0 . For sufficiently small r , the ML estimate gives

$$\left| \int_{C_r} q(z) dz \right| \leq 2\pi r M \rightarrow 0.$$

The left side of (2.2) is independent of r , so $\int_{\gamma} q = 0$. Therefore

$$\int_{\gamma} \frac{f(z)}{z - z_0} dz = f(z_0) \int_{\gamma} \frac{dz}{z - z_0}.$$

Apply the same contour deformation to $1/(z - z_0)$ and parametrize C_r :

$$\int_{\gamma} \frac{dz}{z - z_0} = \int_{C_r} \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{ir e^{i\theta}}{r e^{i\theta}} d\theta = 2\pi i.$$

This gives (2.1). □

Remark. The deformation above is a deformation through *closed curves*: their common start and end point is allowed to move. Below we first define homotopy with fixed endpoints; the closed-curve version and invariance of integrals will be treated in the next lecture.

3 Homotopy of curves

3.1 Continuous deformation with fixed endpoints

Definition 3.1. Let $\gamma, \eta: [a, b] \rightarrow U$ be continuous curves with the same endpoints:

$$\gamma(a) = \eta(a), \quad \gamma(b) = \eta(b).$$

A *homotopy with fixed endpoints* from γ to η in U is a continuous map

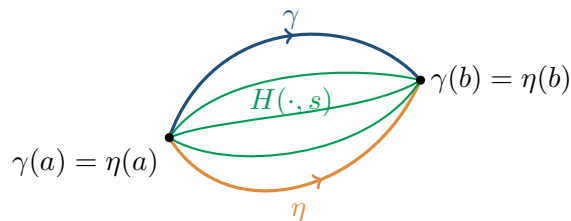
$$H: [a, b] \times [0, 1] \longrightarrow U$$

satisfying

$$\begin{aligned} H(t, 0) &= \gamma(t), & H(t, 1) &= \eta(t) & (a \leq t \leq b), \\ H(a, s) &= \gamma(a) = \eta(a), & H(b, s) &= \gamma(b) = \eta(b) & (0 \leq s \leq 1). \end{aligned}$$

When such an H exists, the curves are *homotopic with fixed endpoints in U* .

For each fixed s , the map $t \mapsto H(t, s)$ is a continuous curve with the same endpoints. As s runs from 0 to 1, these curves deform γ into η , staying inside U throughout.



3.2 Example: curves in a convex set

Definition 3.2. A set $S \subseteq \mathbb{C}$ is *convex* if, for every $z, w \in S$, the line segment

$$[z, w] = \{(1 - s)z + sw : 0 \leq s \leq 1\}$$

lies in S .

Discs, half-planes, and filled triangles are convex. A region with an inward notch need not be convex: a segment joining two of its points can pass outside the region.

Example 3.3 (Straight-line homotopy). Let S be convex, and let $\gamma, \eta: [a, b] \rightarrow S$ be continuous curves with the same endpoints. Define

$$H(t, s) = (1 - s)\gamma(t) + s\eta(t). \quad (3.1)$$

This map is continuous, and convexity ensures that $H(t, s) \in S$. Moreover,

$$H(t, 0) = \gamma(t), \quad H(t, 1) = \eta(t),$$

and the endpoints stay fixed because the endpoints of γ and η agree. Thus any two such curves in a convex set are homotopic with fixed endpoints.

In particular, every curve in a convex set can be deformed, keeping its endpoints fixed, to the straight segment joining those endpoints. The next step is to prove that integrating a holomorphic function is invariant under homotopy, and then to justify the contour deformation used in the Cauchy integral formula.