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We continue Lecture 14's applications of Cauchy's theorem with an indented contour. We then prepare to define integrals of holomorphic functions along continuous curves by establishing the compactness and chain-of-discs results needed to use local primitives.

1 An indented semicircular contour

Continuing the real-variable integral examples from Lecture 14, we compute

$$\boxed{\int_0^\infty \frac{1 - \cos x}{x^2} dx = \frac{\pi}{2}.} \quad (1.1)$$

The integral converges absolutely: near 0, $(1 - \cos x)/x^2 \rightarrow 1/2$, and for $x \geq 1$ its absolute value is at most $2/x^2$.

To evaluate it using a contour integral, set

$$f(z) = \frac{1 - e^{iz}}{z^2} \quad (z \neq 0).$$

This is holomorphic away from 0, and for real $x \neq 0$,

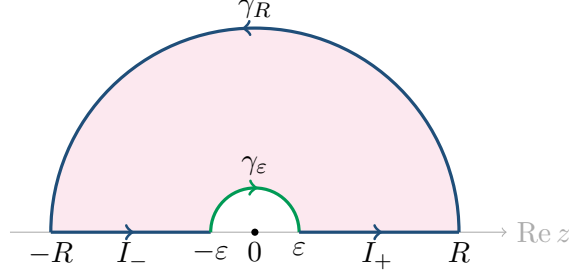
$$f(x) = \frac{1 - \cos x}{x^2} - i \frac{\sin x}{x^2}.$$

We will integrate around a large upper semicircle, making a small semicircular indentation to avoid the origin.

1.1 The contour and the vanishing integral

Fix $0 < \varepsilon < R$. The contour Γ consists, in order, of

- $I_- = [-R, -\varepsilon]$, traversed from left to right;
- γ_ε , the upper semicircle from $-\varepsilon$ to ε , traversed clockwise;
- $I_+ = [\varepsilon, R]$, traversed from left to right;
- $\gamma_R(\theta) = Re^{i\theta}$, $0 \leq \theta \leq \pi$, traversed counterclockwise.



For this contour we have

$$\int_{I_-} f + \int_{\gamma_\epsilon} f + \int_{I_+} f + \int_{\gamma_R} f = 0. \quad (1.2)$$

Here is a direct justification using facts already proved, without assuming a general version of Cauchy's theorem for arbitrary regions. The exponential series gives

$$f(z) = -\frac{i}{z} + E(z), \quad E(z) = -\sum_{n=2}^{\infty} \frac{i^n z^{n-2}}{n!} = \frac{1}{2} + \frac{iz}{6} - \frac{z^2}{24} + \cdots. \quad (1.3)$$

The series for E converges everywhere. Integrating it term by term gives an entire primitive of E , so $\int_{\Gamma} E = 0$. Also,

$$\int_{I_-} \frac{dz}{z} + \int_{I_+} \frac{dz}{z} = \log \frac{\epsilon}{R} + \log \frac{R}{\epsilon} = 0,$$

while the two semicircles contribute

$$\int_{\gamma_\epsilon} \frac{dz}{z} = \int_{\pi}^0 i d\theta = -i\pi, \quad \int_{\gamma_R} \frac{dz}{z} = \int_0^{\pi} i d\theta = i\pi.$$

Thus $\int_{\Gamma} dz/z = 0$, and (1.3) proves (1.2).

1.2 Estimating the arcs and taking limits

The large semicircle. In the closed upper half-plane, $|e^{iz}| = e^{-\text{Im } z} \leq 1$, so $|1 - e^{iz}| \leq 2$. On γ_R , therefore, $|f(z)| \leq 2/R^2$. The ML estimate gives

$$\left| \int_{\gamma_R} f \right| \leq \frac{2}{R^2} \pi R = \frac{2\pi}{R} \rightarrow 0 \quad (R \rightarrow \infty). \quad (1.4)$$

The small semicircle. Use (1.3). Since γ_ϵ goes clockwise, its angle decreases from π to 0:

$$\begin{aligned} \int_{\gamma_\epsilon} f(z) dz &= \int_{\pi}^0 \frac{-i}{\epsilon e^{i\theta}} i \epsilon e^{i\theta} d\theta + \int_{\gamma_\epsilon} E(z) dz \\ &= -\pi + \int_{\gamma_\epsilon} E(z) dz. \end{aligned}$$

Since E is entire, it is continuous on the compact closed unit disc. Hence there is a constant M such that $|E(z)| \leq M$ for $|z| \leq 1$. For $0 < \epsilon \leq 1$,

$$\left| \int_{\gamma_\epsilon} E(z) dz \right| \leq M\pi\epsilon \rightarrow 0.$$

Consequently,

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\gamma_\varepsilon} f = -\pi. \quad (1.5)$$

The real intervals. Their sum is

$$\begin{aligned} \int_{I_-} f + \int_{I_+} f &= \int_{-R}^{-\varepsilon} \frac{1 - e^{ix}}{x^2} dx + \int_{\varepsilon}^R \frac{1 - e^{ix}}{x^2} dx \\ &= 2 \int_{\varepsilon}^R \frac{1 - \cos x}{x^2} dx. \end{aligned}$$

The real parts add because $(1 - \cos x)/x^2$ is even, and the imaginary parts cancel because $\sin x/x^2$ is odd. Substitution into (1.2) yields

$$2 \int_{\varepsilon}^R \frac{1 - \cos x}{x^2} dx = - \int_{\gamma_\varepsilon} f - \int_{\gamma_R} f.$$

First let $R \rightarrow \infty$ using (1.4), and then let $\varepsilon \rightarrow 0^+$ using (1.5). We obtain

$$2 \int_0^\infty \frac{1 - \cos x}{x^2} dx = \pi,$$

which proves (1.1).

2 Integrating holomorphic functions along continuous curves

So far, we have defined contour integrals for piecewise continuously differentiable paths. We now extend the definition to an arbitrary *continuous* curve $\gamma: [a, b] \rightarrow U$, provided that the integrand f is holomorphic on the open set U .

The idea is to use local primitives. If f has a primitive g on all of U , endpoint evaluation suggests the definition

$$\int_{\gamma} f(z) dz = g(\gamma(b)) - g(\gamma(a)).$$

In general, a global primitive need not exist. We instead divide the curve into finitely many pieces, each lying in a disc on which f has a primitive, and add the corresponding endpoint differences.

2.1 On compact sets

For our purposes, a subset $K \subseteq \mathbb{C}$ is **compact** if it is closed and bounded. Recall that $U \subseteq \mathbb{C}$ is **open** if, for every $z \in U$, there is an $r > 0$ such that $D(z, r) \subseteq U$.

Lemma 2.1. *Let $K \subseteq \mathbb{C}$ be compact. Any collection $\{U_\alpha\}_{\alpha \in I}$ of open sets whose union contains K (an open cover of K) has a finite subcollection whose union still contains K . In other words, there are $\alpha_1, \dots, \alpha_n \in I$ such that*

$$K \subseteq U_{\alpha_1} \cup \dots \cup U_{\alpha_n}.$$

The subcollection may be empty when $K = \emptyset$.

Proof. If K is empty, the empty subcollection suffices. Otherwise, suppose that no finite subcollection covers K . Since K is bounded, it is contained in a closed square Q_0 . Divide Q_0 into four equal closed squares. For at least one of these squares, say Q_1 , the set $K \cap Q_1$ cannot be covered by finitely many of the U_α ; otherwise, combining four finite subcollections would cover K .

Repeat this construction to obtain nested closed squares

$$Q_0 \supseteq Q_1 \supseteq Q_2 \supseteq \cdots, \quad \text{diam}(Q_n) = 2^{-n} \text{diam}(Q_0),$$

such that each $K \cap Q_n$ has no finite subcover. In particular, each intersection is nonempty, so choose $z_n \in K \cap Q_n$. For $m \geq n$, both z_m and z_n lie in Q_n , and hence $|z_m - z_n| \leq \text{diam}(Q_n)$. Thus (z_n) is Cauchy and converges to some $z \in \mathbb{C}$. Since K is closed, $z \in K$. Also, each Q_n is closed and contains the tail of the sequence, so $z \in Q_n$ for every n .

Choose $\alpha \in I$ with $z \in U_\alpha$. Since U_α is open, there is an $r > 0$ such that $D(z, r) \subseteq U_\alpha$. For sufficiently large n , $\text{diam}(Q_n) < r$. As $z \in Q_n$, this implies

$$K \cap Q_n \subseteq Q_n \subseteq D(z, r) \subseteq U_\alpha.$$

Thus a single member of the cover covers $K \cap Q_n$, a contradiction. $\square$

Exercise 2.2. Suppose $\gamma: [a, b] \rightarrow \mathbb{C}$ is continuous. Show that it has compact image.

2.2 A uniform neighbourhood of a compact set

Lemma 2.3. Let $U \subseteq \mathbb{C}$ be open and $K \subseteq U$ compact. There is an $r > 0$ such that $D(z, r) \subseteq U$ for every $z \in K$, where $D(z, r) = \{w \in \mathbb{C} : |w - z| < r\}$.

Informally, a compact subset of an open set stays a *uniform* positive distance away from the complement of the open set.

Proof. If K is empty, any $r > 0$ works. Otherwise, for each $z \in K$, openness of U gives a number $r_z > 0$ such that $D(z, 2r_z) \subseteq U$. The smaller discs $\{D(z, r_z) : z \in K\}$ form an open cover of K . By Lemma 2.1, finitely many suffice:

$$K \subseteq \bigcup_{j=1}^n D(z_j, r_{z_j}).$$

Set $r = \min\{r_{z_1}, \dots, r_{z_n}\} > 0$.

Given $z \in K$, choose j with $|z - z_j| < r_{z_j}$. If $w \in D(z, r)$, the triangle inequality gives

$$|w - z_j| \leq |w - z| + |z - z_j| < r + r_{z_j} \leq 2r_{z_j}.$$

Hence $w \in D(z_j, 2r_{z_j}) \subseteq U$. Thus $D(z, r) \subseteq U$ for every $z \in K$, as required. $\square$

Lemma 2.4 (A chain of discs along a curve). Let $\gamma: [a, b] \rightarrow U$ be continuous, with U open and $a < b$. There exist a subdivision

$$a = a_0 < a_1 < \cdots < a_n = b$$

and open discs $D_0, \dots, D_{n-1} \subseteq U$ such that $\gamma([a_j, a_{j+1}]) \subseteq D_j$ for every j .

Proof. The image $K = \gamma([a, b])$ is compact. By Lemma 2.3, choose $r > 0$ such that $D(z, r) \subseteq U$ for every $z \in K$. Also, γ is uniformly continuous on $[a, b]$. Thus there is $\delta > 0$ such that

$$s, t \in [a, b], \quad |s - t| < \delta \quad \implies \quad |\gamma(s) - \gamma(t)| < r/2.$$

Choose a subdivision with $a_{j+1} - a_j < \delta$, and set $D_j = D(\gamma(a_j), r)$. For $t \in [a_j, a_{j+1}]$, $|\gamma(t) - \gamma(a_j)| < r/2$, so the whole subcurve lies in D_j . $\square$

