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1 Review of Lecture 13: primitives and Cauchy's theorem

Throughout this lecture, a *path* means a continuous, piecewise continuously differentiable curve.

Primitives and endpoint evaluation. A primitive of a continuous function f on an open set U is a holomorphic function g on U with $g' = f$. If $\gamma: [a, b] \rightarrow U$ is a path, the chain rule gives

$$\int_{\gamma} f(z) dz = g(\gamma(b)) - g(\gamma(a)). \quad (1.1)$$

Thus the integral depends only on the endpoints, and it is zero when γ is closed. Conversely, on an open connected set, if every closed path integral of a continuous f vanishes, then $g(z) = \int_{z_0}^z f(w) dw$ is well-defined and is a primitive of f .

Primitives on a disc. In Lecture 13 we showed that if f is continuous on a disc D and $\int_{\partial R} f = 0$ for every rectangle $R \subseteq D$, then f has a primitive on D . We constructed it by integrating from the centre along a horizontal and a vertical segment. The rectangle condition makes the two orders agree, and the ML¹ estimate gives $g' = f$.

Theorem 1.1 (Cauchy's theorem on a disc). *Let $D \subseteq \mathbb{C}$ be an open disc and let f be holomorphic on D . Then f has a primitive on D , and for every closed path γ in D ,*

$$\boxed{\int_{\gamma} f(z) dz = 0.} \quad (1.2)$$

Proof. Goursat's theorem gives $\int_{\partial R} f = 0$ for every rectangle $R \subseteq D$. The primitive construction above applies, and (1.1) gives zero because the endpoints agree. $\square$

In particular, if f is *entire*, its integral over every closed path is zero: the compact image of the path lies in some disc. The domain matters: $1/z$ is holomorphic on $\mathbb{C} \setminus \{0\}$, but its integral around the counterclockwise unit circle is $2\pi i$. Local primitives need not give a primitive on the whole domain.

Today we combine Cauchy's theorem with the ML estimate to compute the Fourier transform of the Gaussian by shifting a rectangular contour.

¹This is the estimate $|\int_{\gamma} f| \leq \max_{z \in \gamma} |f(z)| L_{\gamma}$ where L_{γ} is the arclength of γ . The "M" stands for max, and "L" for arclength

2 The Gaussian integral and its Fourier transform

2.1 The Gaussian integral

First recall the real-variable calculation

$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1. \quad (2.1)$$

Let I denote this integral. It converges: for $|x| \geq 1$, $e^{-\pi x^2} \leq e^{-\pi|x|}$. Since the integrand is nonnegative, we may write the product as a double integral and use polar coordinates:

$$\begin{aligned} I^2 &= \int_{\mathbb{R}^2} e^{-\pi(x^2+y^2)} dx dy \\ &= \int_0^{2\pi} \int_0^{\infty} e^{-\pi r^2} r dr d\theta = 2\pi \left[-\frac{e^{-\pi r^2}}{2\pi} \right]_0^{\infty} = 1. \end{aligned}$$

As $I > 0$, we obtain $I = 1$.

2.2 A family of oscillatory integrals

For a fixed $\xi \in \mathbb{R}$, consider

$$F(\xi) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} dx.$$

This integral converges absolutely, since $|e^{-2\pi i x \xi}| = 1$. When $\xi = 0$, it is (2.1). The general answer is again a Gaussian.

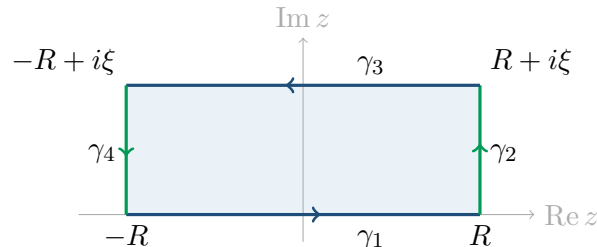
Theorem 2.1. *For every $\xi \in \mathbb{R}$,*

$$\boxed{\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} dx = e^{-\pi \xi^2}.} \quad (2.2)$$

2.3 Shifting the contour

Proof of Theorem 2.1. The substitution $x \mapsto -x$ gives $F(-\xi) = F(\xi)$, so it is enough to consider $\xi > 0$; the case $\xi = 0$ is already known.

Let $f(z) = e^{-\pi z^2}$, which is entire. For $R > 0$, take the rectangular contour with vertices $-R$, R , $R + i\xi$, and $-R + i\xi$, oriented counterclockwise.



Cauchy's theorem gives

$$\int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\gamma_3} f + \int_{\gamma_4} f = 0. \quad (2.3)$$

We may parametrize the four sides by

$$\begin{aligned} \gamma_1(t) &= t, & -R \leq t \leq R, & \gamma_1'(t) = 1, \\ \gamma_2(t) &= R + it, & 0 \leq t \leq \xi, & \gamma_2'(t) = i, \\ \gamma_3(t) &= -t + i\xi, & -R \leq t \leq R, & \gamma_3'(t) = -1, \\ \gamma_4(t) &= -R + i(\xi - t), & 0 \leq t \leq \xi, & \gamma_4'(t) = -i. \end{aligned}$$

The bottom side. By (2.1),

$$\int_{\gamma_1} f = \int_{-R}^R e^{-\pi t^2} dt \longrightarrow 1 \quad (R \rightarrow \infty).$$

The vertical sides. On $z = R + it$,

$$(R + it)^2 = R^2 + 2iRt - t^2, \quad |e^{-\pi(R+it)^2}| = e^{-\pi R^2} e^{\pi t^2}.$$

Hence

$$\left| \int_{\gamma_2} f \right| \leq e^{-\pi R^2} \int_0^\xi e^{\pi t^2} dt \leq \xi e^{\pi \xi^2} e^{-\pi R^2} \longrightarrow 0.$$

Exactly the same estimate holds on the left side γ_4 .

The top side. Its orientation is from right to left. Using the parametrization above,

$$\begin{aligned} \int_{\gamma_3} f &= - \int_{-R}^R e^{-\pi(-t+i\xi)^2} dt \\ &= -e^{\pi \xi^2} \int_{-R}^R e^{-\pi t^2} e^{2\pi i t \xi} dt \\ &= -e^{\pi \xi^2} \int_{-R}^R e^{-\pi x^2} e^{-2\pi i x \xi} dx, \end{aligned}$$

where the last equality uses $x = -t$. Absolute convergence gives

$$\int_{\gamma_3} f \longrightarrow -e^{\pi \xi^2} F(\xi).$$

Taking $R \rightarrow \infty$ in (2.3), we obtain

$$1 + 0 - e^{\pi \xi^2} F(\xi) + 0 = 0.$$

Therefore $F(\xi) = e^{-\pi \xi^2}$, as claimed. $\square$

Corollary 2.2 (Real and imaginary parts). *For every $\xi \in \mathbb{R}$,*

$$\int_{-\infty}^{\infty} e^{-\pi x^2} \cos(2\pi x \xi) dx = e^{-\pi \xi^2}, \quad (2.4)$$

$$\int_{-\infty}^{\infty} e^{-\pi x^2} \sin(2\pi x \xi) dx = 0. \quad (2.5)$$

The second identity also follows directly because its integrand is odd. For the first, the contour shift turns the oscillatory factor into the constant factor $e^{\pi \xi^2}$ on the top edge of the rectangle.

Remark (What made the estimate work?). The height ξ was fixed while $R \rightarrow \infty$. Thus the lengths of the vertical sides stayed fixed, and the factor $e^{-\pi R^2}$ forced their integrals to vanish. The factor on the top side is $e^{+\pi \xi^2}$; solving for $F(\xi)$ produces $e^{-\pi \xi^2}$.