

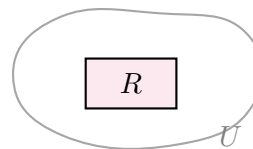
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1 Where we are

Last time we proved Goursat's theorem: if f is holomorphic on an open set U and $R \subseteq U$ is a rectangle, then

$$\int_{\partial R} f = 0. \quad (1.1)$$



This has many consequences:

- *theoretical*: existence of primitives, Cauchy's theorem on a disc (today);
- *practical*: computation of *real* integrals, such as

$$\int_0^\infty \frac{1 - \cos x}{x^2} dx = \frac{\pi}{2} \quad (1.2)$$

(next time).

2 Primitives

Definition 2.1 (Primitive). Let f be a continuous function on an open set $U \subseteq \mathbb{C}$. A *primitive* for f (or *holomorphic primitive*) is a function $g: U \rightarrow \mathbb{C}$ such that g is holomorphic and

$$g' = f \quad \text{on } U. \quad (2.1)$$

Example 2.2 (Analytic functions have local primitives). Recall: if f is analytic on U , so

$$f(z) = \sum a_n(z - z_0)^n \quad \text{for } z \text{ near } z_0, \quad (2.2)$$

then $f'(z)$ exists and is computed term by term. But we can also “integrate term by term” to get the series

$$g(z) = \sum_n \frac{a_n}{n+1} (z - z_0)^{n+1}. \quad (2.3)$$

What do we know about this series? Its coefficient of $(z - z_0)^{n+1}$ is $\frac{a_n}{n+1}$, and

$$\limsup_n \left| \frac{a_n}{n+1} \right|^{1/n} = \limsup_n \frac{|a_n|^{1/n}}{(n+1)^{1/n}} = \limsup_n |a_n|^{1/n}, \quad (2.4)$$

since $(n+1)^{1/n} \rightarrow 1$. (Shifting the index by one does not change the radius of convergence.) Therefore $g(z)$ is also convergent for z near z_0 , so

$$g(z) \text{ analytic} \implies \text{holomorphic, and } g' = f \quad (2.5)$$

by term-by-term differentiation. So g is a primitive for f on a disc around z_0 .

Some remarks on primitives

Proposition 2.3. *If you have a primitive for f , then path integrals are easy to compute. Let f be continuous on U with primitive g , and $\gamma: [a, b] \rightarrow U$ a path. Then*

$$\int_{\gamma} f = g(\gamma(b)) - g(\gamma(a)). \quad (2.6)$$

Proof. Since $g' = f$ (g is a primitive for f), the chain rule gives

$$\frac{d}{dt} g(\gamma(t)) = g'(\gamma(t)) \gamma'(t) = f(\gamma(t)) \gamma'(t). \quad (2.7)$$

Therefore, by FTC,

$$\int_{\gamma} f = \int_a^b \frac{d}{dt} g(\gamma(t)) dt = g(\gamma(b)) - g(\gamma(a)). \quad \square \quad (2.8)$$

So the answer is computed in terms of the primitive and the endpoints of the curve γ .

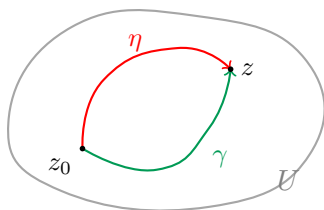
Exercise 2.4. Justify the chain rule used above: if g is holomorphic and γ is differentiable at t , then $\frac{d}{dt} g(\gamma(t)) = g'(\gamma(t)) \gamma'(t)$. *Hint.* Write $g(w) - g(w_0) = (g'(w_0) + \varphi(w))(w - w_0)$ with $\varphi(w) \rightarrow 0$ as $w \rightarrow w_0$, and take $w = \gamma(t + s)$, $w_0 = \gamma(t)$.

Corollary 2.5. *If f admits a primitive on U and γ is a closed path in U ($\gamma(a) = \gamma(b)$), then $\int_{\gamma} f = 0$.*

Warning. Suppose f is analytic on U and γ is a closed path in U . Is $\int_{\gamma} f = 0$? Example 2.2 only gives a primitive *near each point*, not necessarily one on all of U . Indeed, for $f(z) = \frac{1}{z}$ on $U = \mathbb{C} \setminus \{0\}$ and γ the unit circle, we computed $\int_{\gamma} f = 2\pi i \neq 0$ in Lecture 11. So $\frac{1}{z}$ has no primitive on $\mathbb{C} \setminus \{0\}$, even though it is analytic there.

Proposition 2.6. *Let $U \subseteq \mathbb{C}$ be open and connected, and $f: U \rightarrow \mathbb{C}$ continuous. If $\int_{\gamma} f = 0$ for all closed paths γ in U , then f has a primitive.*

Why? We need to construct a holomorphic primitive to f .



Idea. Fix $z_0 \in U$. For $z \in U$, choose any path γ connecting z_0 and z (possible since U is connected), and set

$$g(z) = \int_{\gamma} f. \quad (2.9)$$

Why is this a well-defined function, i.e. independent of the path γ ? If η is another path from z_0 to z , then $\gamma + \eta^-$ (go along γ , then back along η reversed) is a closed loop, so

$$0 = \int_{\gamma + \eta^-} f = \int_{\gamma} f + \int_{\eta^-} f = \int_{\gamma} f - \int_{\eta} f, \quad \therefore \int_{\gamma} f = \int_{\eta} f. \quad (2.10)$$

So g is well-defined. But is it holomorphic with $g' = f$? We will prove a similar statement below; the same argument works here (Exercise).

3 Primitives on a disc

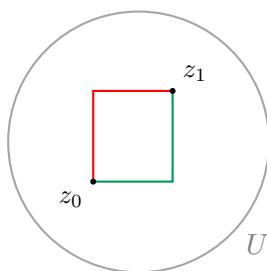
Theorem 3.1. *Let U be an open disc centred at $z_0 \in \mathbb{C}$, and f a continuous function on U . Assume*

$$\int_{\partial R} f = 0 \quad \text{for any rectangle } R \subseteq U. \quad (3.1)$$

For $z_1 \in U$ define

$$g(z_1) = \int_{z_0}^{z_1} f, \quad (3.2)$$

where we choose any rectangular path connecting z_0 and z_1 (i.e. one horizontal and one vertical segment). Then g is a well-defined holomorphic primitive for f .



Note that the rectangle with opposite corners z_0 and z_1 lies in U : its farthest point from the centre z_0 is z_1 . So both rectangular paths lie in U .

Corollary 3.2 (Cauchy's theorem on a disc). *If f is holomorphic on a disc U , then*

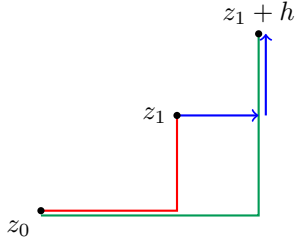
$$\int_{\gamma} f = 0 \quad \text{for any closed path } \gamma \text{ in } U. \quad (3.3)$$

Proof. By Goursat's theorem, $\int_{\partial R} f = 0$ for every rectangle $R \subseteq U$; and f is continuous since it is holomorphic. By Theorem 3.1, f has a primitive on U , so $\int_{\gamma} f = 0$ by Corollary 2.5. $\square$

Proof of Theorem 3.1. 1) Well-definedness. Same idea as in Proposition 2.6: the two rectangular paths from z_0 to z_1 (horizontal-then-vertical, and vertical-then-horizontal) together form $\pm\partial R$, where R is the rectangle with opposite corners z_0, z_1 . Since $\int_{\partial R} f = 0$, the two paths give the same integral. (If z_0 and z_1 share a real or imaginary part, there is only one such path.) From now on we compute g using the horizontal-then-vertical path.

2) g is holomorphic and $g' = f$. Given $z_1 \in U$, we want to show

$$\lim_{h \rightarrow 0} \frac{g(z_1 + h) - g(z_1)}{h} = f(z_1). \quad (3.4)$$



We claim that $g(z_1 + h) - g(z_1)$ is the integral of f over the blue path from z_1 : horizontally, then vertically to $z_1 + h$.

$$g(z_1 + h) - g(z_1) = \int_{\text{green}} f - \int_{\text{red}} f = \int_{z_1}^{z_1+h} f. \quad (3.5)$$

Indeed, the green path minus the red path minus the blue path is (up to sign) the boundary of the rectangle with corners $\text{Re } z_1 + i \text{Im } z_0$, $\text{Re}(z_1 + h) + i \text{Im } z_0$, $\text{Re}(z_1 + h) + i \text{Im } z_1$ and z_1 . For h small this rectangle lies in the disc U (all four corners do, and a disc is convex), so its boundary integral vanishes by hypothesis.

Now

$$\frac{g(z_1 + h) - g(z_1)}{h} = \frac{1}{h} \int_{z_1}^{z_1+h} f(z) dz \quad (\text{over the blue path}). \quad (3.6)$$

Since f is continuous, there is $\psi(z)$ with

$$f(z) = f(z_1) + \psi(z) \quad \text{and} \quad \lim_{z \rightarrow z_1} \psi(z) = 0. \quad (3.7)$$

Plugging this in,

$$\frac{g(z_1 + h) - g(z_1)}{h} = \frac{1}{h} \left[\int_{z_1}^{z_1+h} f(z_1) dz + \int_{z_1}^{z_1+h} \psi(z) dz \right]. \quad (3.8)$$

The constant $f(z_1)$ has primitive $f(z_1)z$, so by Proposition 2.3

$$\frac{1}{h} \int_{z_1}^{z_1+h} f(z_1) dz = \frac{1}{h} f(z_1)((z_1 + h) - z_1) = f(z_1). \quad (3.9)$$

For the second term, by the ML-estimate,

$$\left| \frac{1}{h} \int_{z_1}^{z_1+h} \psi(z) dz \right| \leq \max_{\text{blue}} |\psi(z)| \cdot \frac{1}{|h|} \cdot \text{arclength}(\text{blue}) \leq 2 \max_{\text{blue}} |\psi(z)|, \quad (3.10)$$

since the blue path has length $|\operatorname{Re} h| + |\operatorname{Im} h| \leq 2|h|$. As $h \rightarrow 0$, the blue path shrinks to z_1 , so $\max_{\text{blue}} |\psi| \rightarrow 0$. Therefore

$$\lim_{h \rightarrow 0} \frac{g(z_1 + h) - g(z_1)}{h} = f(z_1), \quad (3.11)$$

i.e. g is holomorphic on U with $g' = f$. □