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1 Review of path integrals and estimates

Recall the following definitions and estimates from Lecture 11. Let $U \subseteq \mathbb{C}$ be open.

Definition 1.1 (Piecewise continuously differentiable path). A curve $\gamma: [a, b] \rightarrow U$ is *piecewise continuously differentiable* if it is continuous on $[a, b]$ and there is a finite partition $a = t_0 < t_1 < \cdots < t_m = b$ such that γ is continuously differentiable on each $[t_{j-1}, t_j]$, with one-sided derivatives at the endpoints. Unless said otherwise, a *path* means such a curve. Its derivative need not exist at the joining points $t_1, \dots, t_{m-1}$.

Definition 1.2 (Path integral). For f continuous on U and $\gamma: [a, b] \rightarrow U$ a path, write $\gamma(t) = u(t) + iv(t)$, so $\gamma'(t) = u'(t) + iv'(t)$ on each C^1 piece. Then

$$\int_{\gamma} f = \int_{\gamma} f dz := \int_a^b f(\gamma(t)) \gamma'(t) dt. \quad (1.1)$$

Proposition 1.3 (Triangle inequality). For continuous $F: [a, b] \rightarrow \mathbb{C}$,

$$\left| \int_a^b F(t) dt \right| \leq \int_a^b |F(t)| dt. \quad (1.2)$$

Proposition 1.4 (ML estimate). For a path $\gamma: [a, b] \rightarrow \mathbb{C}$, let

$$L(\gamma) = \text{arclength of } \gamma = \int_a^b |\gamma'(t)| dt. \quad (1.3)$$

Then

$$\left| \int_{\gamma} f \right| \leq \left(\sup_{z \in \gamma([a, b])} |f(z)| \right) \cdot L(\gamma). \quad (1.4)$$

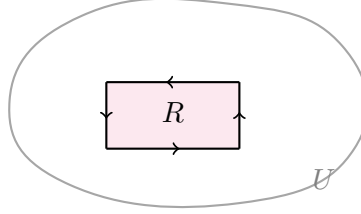
Proof. Why? Since $\int_{\gamma} f = \int_a^b f(\gamma(t)) \gamma'(t) dt$, the basic estimate above gives

$$\left| \int_{\gamma} f \right| \leq \int_a^b |f(\gamma(t))| |\gamma'(t)| dt \leq \sup_{z \in \gamma([a, b])} |f(z)| \int_a^b |\gamma'(t)| dt. \quad (1.5)$$

□

2 The Cauchy–Goursat theorem

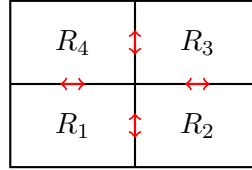
Let $U \subseteq \mathbb{C}$ be open and f holomorphic on U . Let R be any (closed) rectangle in U with sides parallel to the axes, and write ∂R for its boundary, traversed counterclockwise.



Theorem 2.1 (Cauchy–Goursat). $\int_{\partial R} f = 0$.

Proof. There are three steps. We don't use that f is holomorphic in Step 1.

Step 1: subdivide. Cut R into four congruent subrectangles R_1, R_2, R_3, R_4 by bisecting its sides, and orient each boundary counterclockwise.



Each interior edge is traversed twice, in opposite directions, so those contributions cancel and

$$\int_{\partial R} f = \int_{\partial R_1} f + \int_{\partial R_2} f + \int_{\partial R_3} f + \int_{\partial R_4} f. \quad (2.1)$$

Hence

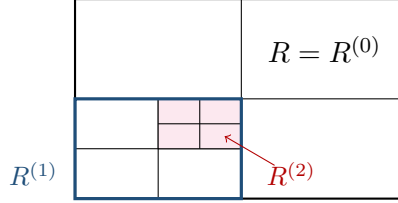
$$\left| \int_{\partial R} f \right| \leq \sum_{i=1}^4 \left| \int_{\partial R_i} f \right|, \quad (2.2)$$

so there exists some subrectangle $R^{(1)} \in \{R_1, \dots, R_4\}$ such that

$$\left| \int_{\partial R^{(1)}} f \right| \geq \frac{1}{4} \left| \int_{\partial R} f \right|. \quad (2.3)$$

Apply the same reasoning to $\int_{\partial R^{(1)}} f$, i.e. break up $R^{(1)}$ into four subrectangles, and argue that there is one of them, $R^{(2)}$, with

$$\left| \int_{\partial R^{(2)}} f \right| \geq \frac{1}{4} \left| \int_{\partial R^{(1)}} f \right| \implies \left| \int_{\partial R^{(2)}} f \right| \geq \frac{1}{4} \cdot \frac{1}{4} \left| \int_{\partial R} f \right|. \quad (2.4)$$



Continue to find rectangles $R = R^{(0)} \supset R^{(1)} \supset R^{(2)} \supset \dots$ such that

$$\left| \int_{\partial R^{(n)}} f \right| \geq \frac{1}{4^n} \left| \int_{\partial R} f \right|. \quad (2.5)$$

Step 2: the rectangles shrink to a point. Let

$$L_n = \text{arclength of } \partial R^{(n)} \text{ (the perimeter of this rectangle)}, \quad d_n = \text{diam}(R^{(n)}), \quad (2.6)$$

where the diameter is the maximum distance between two points of $R^{(n)}$. Since each step halves the side lengths, we find

$$L_n = \frac{1}{2^n} L_0, \quad d_n = \frac{1}{2^n} d_0. \quad (2.7)$$

Claim. $\bigcap_{n=1}^{\infty} R^{(n)}$ consists of a single point.

Since $d_n \rightarrow 0$, there cannot be two distinct points in $\bigcap_n R^{(n)}$. But why is there one point? Pick $\alpha_j \in R^{(j)}$. For $j, k \geq n$ both α_j, α_k lie in $R^{(n)}$, so $|\alpha_j - \alpha_k| \leq d_n \rightarrow 0$: the sequence $\{\alpha_j\}$ is Cauchy. Therefore it has a limit z_0 . For each n , the tail $\{\alpha_j\}_{j \geq n}$ lies in $R^{(n)}$, which is closed (it contains the limit of any Cauchy sequence it contains), so $z_0 \in R^{(n)}$. Thus $z_0 \in \bigcap_n R^{(n)}$.

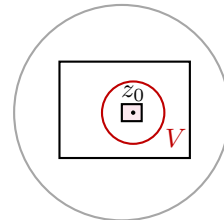
Step 3: use holomorphicity. Now we want to input that f is holomorphic. We pick z_0 to be the point $\bigcap_{n=1}^{\infty} R^{(n)}$. Then f being holomorphic at $z_0 \in U$ means: there is a function $\varphi(z)$ defined on a neighbourhood $V \ni z_0$, with $\lim_{z \rightarrow z_0} \varphi(z) = 0$, $\varphi(z_0) = 0$, and such that

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \varphi(z)(z - z_0) \quad (z \in V).$$

Note that the function φ is continuous on V : this is certainly true at $z = z_0$, and for $z \neq z_0$ it equals $\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)$. But f is continuous, being $\mathbb{C}$ -differentiable.

Since $d_n \rightarrow 0$ and $z_0 \in R^{(n)}$, we may choose n large so that $R^{(n)} \subseteq V$. Then

$$\begin{aligned} \int_{\partial R^{(n)}} f &= \underbrace{\int_{\partial R^{(n)}} f(z_0) dz}_0 + \underbrace{\int_{\partial R^{(n)}} f'(z_0)(z - z_0) dz}_0 \\ &\quad + \int_{\partial R^{(n)}} \varphi(z)(z - z_0) dz, \end{aligned}$$



where the first two integrals vanish because integrals of polynomials over closed paths vanish (Lecture 11).

So

$$\int_{\partial R^{(n)}} f = \int_{\partial R^{(n)}} \varphi(z)(z - z_0) dz \quad \text{for any } n \text{ sufficiently large.} \quad (2.8)$$

Combining with (2.5) and the ML-estimate,

$$\begin{aligned} \left| \int_{\partial R} f \right| &\leq 4^n \left| \int_{\partial R^{(n)}} f \right| \leq 4^n \left| \int_{\partial R^{(n)}} \varphi(z)(z - z_0) dz \right| \\ &\leq 4^n \cdot \sup_{z \in \partial R^{(n)}} |\varphi(z)| \cdot |z - z_0| \cdot L_n \quad (L_n = \text{arclength of } \partial R^{(n)}) \\ &\leq 4^n \cdot \sup_{z \in \partial R^{(n)}} |\varphi(z)| \cdot \sup_{z \in \partial R^{(n)}} |z - z_0| \cdot \frac{1}{2^n} L_0 \\ &\leq 4^n \cdot \text{diam}(R^{(n)}) \cdot \sup_{z \in \partial R^{(n)}} |\varphi(z)| \cdot \frac{1}{2^n} L_0 \quad (z_0 \in R^{(n)}) \\ &\leq 4^n \cdot \frac{1}{2^n} d_0 \cdot \sup_{z \in \partial R^{(n)}} |\varphi(z)| \cdot \frac{1}{2^n} L_0 = C \cdot \sup_{z \in \partial R^{(n)}} |\varphi(z)|, \end{aligned}$$

with $C = d_0 L_0$ independent of n . As $n \rightarrow \infty$, every point of $\partial R^{(n)}$ is within $d_n \rightarrow 0$ of z_0 , and $\varphi(z) \rightarrow 0$ as $|z - z_0| \rightarrow 0$. So the right-hand side tends to 0, and hence $\int_{\partial R} f = 0$. $\square$

Remark. The factor 4^n lost in Step 1 is exactly compensated by the factors $\frac{1}{2^n}$ from the diameter and the perimeter in Step 3. The theorem really uses only that f is $\mathbb{C}$ -differentiable at each point; we did not assume f' is continuous.

Exercise 2.2.