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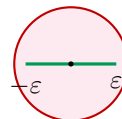
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1 Analytic extensions of real functions

Recall from Lecture 10 that if two convergent power series agree on an infinite set accumulating at 0, they are equal as power series.

Let $f(x)$ be a real-valued function on an interval $(-c, c) \subseteq \mathbb{R}$.

Definition 1.1 (Extension). We call a function g , defined on a neighbourhood of 0 in $\mathbb{C}$, an *extension* of f if $g(x) = f(x)$ for all real $x \in (-\varepsilon, \varepsilon)$, for some $\varepsilon > 0$.



Corollary 1.2. *There is at most one analytic function around 0 which restricts to $f(x)$. More precisely, if g_1 and g_2 are analytic extensions of f , then $g_1 = g_2$ on some disc around 0.*

Proof. Near 0, write $g_1(z) = \sum a_n z^n$ and $g_2(z) = \sum b_n z^n$ with positive radii of convergence. These agree at the points $x = 1/n$ for all large n , which accumulate at 0. By the identity principle from Lecture 10, $\sum a_n T^n = \sum b_n T^n$, so $g_1 = g_2$ on a common disc of convergence. $\square$

Warning. We are not saying that such an extension exists, just that it is unique if it does exist.

Example 1.3 (The exponential). The function e^x defined on $\mathbb{R}$ has at most one analytic extension. Consider

$$\text{Exp}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad (1.1)$$

which has infinite radius of convergence. On the real line it restricts to e^x , by the usual power series expansion of e^x from calculus. So $\text{Exp}(z)$ is *the* analytic extension of e^x .

On the other hand, we previously defined, for $z = x + iy$,

$$e^z = e^x e^{iy} = e^x (\cos y + i \sin y). \quad (1.2)$$

Question. What is the relation between $\text{Exp}(z)$ and e^z ?

Note:

(1) $e^z e^w = e^{z+w}$, and likewise $\text{Exp}(z) \text{Exp}(w) = \text{Exp}(z+w)$. (For the latter, multiply the two absolutely convergent series and use the binomial theorem: the coefficient of the degree- n part of the product is $\sum_k \frac{z^k w^{n-k}}{k!(n-k)!} = \frac{(z+w)^n}{n!}$.)

(2) $\text{Exp}(x) = e^x$ on the real line.

(3) *Check:* $\text{Exp}(iy) = e^{iy}$.

For (3), plug $z = iy$ into the definition and separate even and odd powers (allowed since the series converges absolutely), using $i^{2n} = (-1)^n$ and $i^{2n+1} = (-1)^n i$:

$$\text{Exp}(iy) = \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{y^{2n}}{(2n)!}}_{\cos y} + i \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{(2n+1)!}}_{\sin y}. \quad (1.3)$$

Therefore $\text{Exp}(iy) = \cos y + i \sin y$. Combining (1)–(3),

$$\text{Exp}(x + iy) = \text{Exp}(x) \text{Exp}(iy) = e^x (\cos y + i \sin y) = e^{x+iy}, \quad (1.4)$$

so $\text{Exp}(z) = e^z$ for all $z \in \mathbb{C}$.

2 Path integrals and Cauchy's theorems

Integrating complex-valued functions of a real variable

Let $F: [a, b] \rightarrow \mathbb{C}$, and write $F(t) = u(t) + iv(t)$ with u, v real-valued. We define

$$\int_a^b F(t) dt = \int_a^b u(t) dt + i \int_a^b v(t) dt, \quad (2.1)$$

which takes $\mathbb{C}$ -values. All the familiar rules carry over by working with real and imaginary parts.

Exercise 2.1. Check that if F and G have continuously differentiable real and imaginary parts, then integration by parts holds:

$$\int_a^b F(t) G'(t) dt = [F(t) G(t)]_a^b - \int_a^b G(t) F'(t) dt. \quad (2.2)$$

Also, the “Fundamental Theorem of Calculus” (FTC) holds: if F is continuous, then $s \mapsto \int_a^s F(t) dt$ has derivative $F(s)$ for $s \in (a, b)$ (with one-sided derivatives at the endpoints).

Proposition 2.2 (Basic estimate). *For continuous $F: [a, b] \rightarrow \mathbb{C}$,*

$$\left| \int_a^b F(t) dt \right| \leq \int_a^b |F(t)| dt. \quad (2.3)$$

Proof. Let $I = \int_a^b F(t) dt$ and write $I = |I|e^{i\theta}$. Then

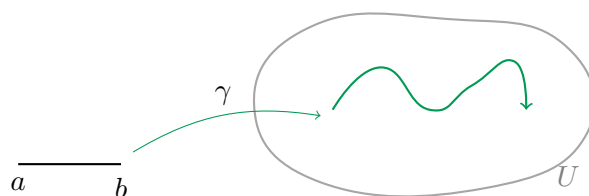
$$|I| = e^{-i\theta} I = \int_a^b e^{-i\theta} F(t) dt = \int_a^b \text{Re}(e^{-i\theta} F(t)) dt \leq \int_a^b |F(t)| dt, \quad (2.4)$$

where the third equality holds because $|I|$ is real, so the imaginary part of the integral vanishes. $\square$

Integrating along a path

More interesting: let $U \subseteq \mathbb{C}$ be open and f a function on U .

Definition 2.3 (Piecewise continuously differentiable path). A curve $\gamma: [a, b] \rightarrow U$ is *piecewise continuously differentiable* if it is continuous on $[a, b]$ and there is a finite partition $a = t_0 < t_1 < \dots < t_m = b$ such that γ is continuously differentiable on each $[t_{j-1}, t_j]$, with one-sided derivatives at the endpoints. Unless said otherwise, a *path* means such a curve. Its derivative need not exist at the joining points $t_1, \dots, t_{m-1}$.



Definition 2.4 (Path integral). For f continuous on U and $\gamma: [a, b] \rightarrow U$ a path, write $\gamma(t) = u(t) + iv(t)$, so $\gamma'(t) = u'(t) + iv'(t)$ on each C^1 piece. Then

$$\int_{\gamma} f = \int_{\gamma} f dz := \int_a^b f(\gamma(t)) \gamma'(t) dt. \quad (2.5)$$

Example 2.5. Let $\gamma(\theta) = \cos \theta + i \sin \theta = e^{i\theta}$ for $\theta \in [0, 2\pi]$, and $f(z) = \frac{1}{z}$. Then $\gamma'(\theta) = ie^{i\theta}$, so

$$\int_{\gamma} f = \int_0^{2\pi} f(\gamma(\theta)) \gamma'(\theta) d\theta = \int_0^{2\pi} \frac{1}{e^{i\theta}} ie^{i\theta} d\theta = 2\pi i. \quad (2.6)$$

Remark.

- (1) *What happens if we change orientation?* For $\gamma(\theta) = e^{-i\theta}$, $\theta \in [0, 2\pi]$, the same computation gives

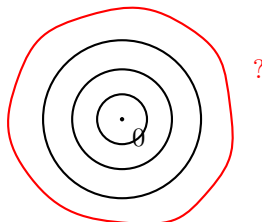
$$\int_{\gamma} \frac{dz}{z} = \int_0^{2\pi} \frac{-ie^{-i\theta}}{e^{-i\theta}} d\theta = -2\pi i. \quad (2.7)$$

So orientation matters.

(2) What if we replace $\gamma(\theta)$ with $\gamma_R(\theta) = Re^{i\theta}$? Then

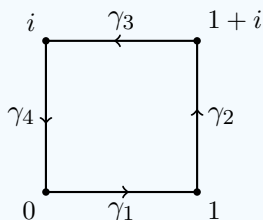
$$\int_{\gamma_R} \frac{dz}{z} = \int_0^{2\pi} \frac{1}{Re^{i\theta}} Rie^{i\theta} d\theta = 2\pi i, \quad (2.8)$$

independently of R . We will see later that the answer is the same for any closed path winding once counterclockwise around 0.



The unit square

Example 2.6. Let $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ be the boundary of the unit square with vertices $0, 1, 1 + i, i$, traversed counterclockwise. What is $\int_{\gamma} 1$? Parametrize each side by $0 \leq t \leq 1$:



Then

	$\gamma_j(t)$	$\gamma'_j(t)$
γ_1	t	1
γ_2	$1 + it$	i
γ_3	$(1 - t) + i$	-1
γ_4	$(1 - t)i$	$-i$

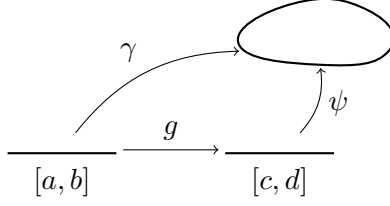
$$\int_{\gamma} 1 = \underbrace{\int_0^1 1 dt}_{\gamma_1} + \underbrace{\int_0^1 i dt}_{\gamma_2} + \underbrace{\int_0^1 (-1) dt}_{\gamma_3} + \underbrace{\int_0^1 (-i) dt}_{\gamma_4} = 0. \quad (2.9)$$

Exercise 2.7. With γ as in Example 2.6, show that $\int_{\gamma} z dz = 0$.

Further properties of path integrals

Proposition 2.8 (Independence of parametrization). *Let $\psi: [c, d] \rightarrow U$ be a path and let $g: [a, b] \rightarrow [c, d]$ be continuously differentiable with $g(a) = c$ and $g(b) = d$. Set $\gamma = \psi \circ g: [a, b] \rightarrow U$, i.e. $\gamma(t) = \psi(g(t))$. Then*

$$\int_{\gamma} f = \int_{\psi} f. \quad (2.10)$$



Proof. By the chain rule $\gamma'(t) = \psi'(g(t))g'(t)$, so substituting $s = g(t)$,

$$\int_{\gamma} f = \int_a^b f(\psi(g(t))) \psi'(g(t)) g'(t) dt = \int_c^d f(\psi(s)) \psi'(s) ds = \int_{\psi} f. \quad (2.11)$$

(Apply the real change-of-variables formula to real and imaginary parts.) $\square$

Proposition 2.9. For a path $\gamma: [a, b] \rightarrow \mathbb{C}$, let

$$L(\gamma) = \text{arclength of } \gamma = \int_a^b |\gamma'(t)| dt. \quad (2.12)$$

Then

$$\left| \int_{\gamma} f \right| \leq \left(\sup_{z \in \gamma([a, b])} |f(z)| \right) \cdot L(\gamma). \quad (2.13)$$

Proof. Why? Since $\int_{\gamma} f = \int_a^b f(\gamma(t)) \gamma'(t) dt$, the basic estimate above gives

$$\left| \int_{\gamma} f \right| \leq \int_a^b |f(\gamma(t))| |\gamma'(t)| dt \leq \sup_{z \in \gamma([a, b])} |f(z)| \int_a^b |\gamma'(t)| dt. \quad (2.14)$$

$\square$

We will also use the following simple observation.

Lemma 2.10. If Q is a polynomial and $\gamma: [a, b] \rightarrow \mathbb{C}$ is a closed path ($\gamma(a) = \gamma(b)$), then $\int_{\gamma} Q(z) dz = 0$. In particular $\int_{\gamma} c dz = 0$ and $\int_{\gamma} (z - z_0) dz = 0$ for any constants $c, z_0 \in \mathbb{C}$.

Proof. By linearity it suffices to take $Q(z) = z^n$ for $n \geq 0$. Choose a partition $a = t_0 < \dots < t_m = b$ on whose pieces γ is C^1 . On each piece the chain rule gives $\frac{d}{dt}(\gamma(t)^{n+1}/(n+1)) = \gamma(t)^n \gamma'(t)$. Applying the Fundamental Theorem of Calculus on each piece and summing, the intermediate endpoint terms cancel:

$$\int_{\gamma} z^n dz = \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \gamma(t)^n \gamma'(t) dt = \frac{\gamma(b)^{n+1} - \gamma(a)^{n+1}}{n+1} = 0. \quad (2.15)$$

The last statement follows by taking $Q(z) = c$ and $Q(z) = z - z_0$. $\square$

We will prove next time:

Theorem 2.11 (Cauchy–Goursat). Let f be holomorphic on an open set U , and let R be a rectangle with $R \subseteq U$. Then

$$\int_{\partial R} f = 0. \quad (2.16)$$