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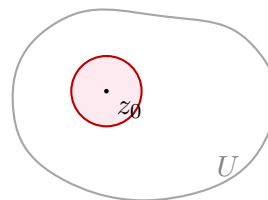
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1 Analytic functions are holomorphic

We begin by recalling the definition from last time.

Definition 1.1 (Analytic). Let $U \subseteq \mathbb{C}$ be open. A function $f: U \rightarrow \mathbb{C}$ is *analytic* if for each $z_0 \in U$ there is a power series $\sum a_n T^n$ with positive radius of convergence such that

$$f(z) = \sum a_n (z - z_0)^n \quad \text{for all } z \text{ near } z_0. \quad (1.1)$$



Remark.

- (1) We have not assumed anything else about f : no continuity, no differentiability. Everything below is a *consequence* of having local power series expansions.
- (2) If $\sum b_n T^n$ is a convergent power series on $D(0, r)$, then the function it defines on $D(0, r)$ is analytic (Lecture 9).
- (3) If $\sum b_n T^n$ has radius of convergence $r > 0$, then $f(z) = \sum b_n z^n$ is $\mathbb{C}$ -differentiable at each $z \in D(0, r)$, and in fact

$$f'(z) = \sum_{n \geq 1} n b_n z^{n-1}, \quad \text{so} \quad f'(0) = b_1. \quad (1.2)$$

That is, f is holomorphic on $D(0, r)$ (Lecture 9).

In fact $f'(z)$ is again given by a power series with radius of convergence r , so it is also holomorphic by the same argument; and so are $f''(z)$, $f'''(z)$, $\dots$. That is, f is infinitely $\mathbb{C}$ -differentiable on $D(0, r)$. Moreover the coefficients can be read off from the derivatives at 0.

Exercise 1.2. Check that $f^{(k)}(0) = k! b_k$ for every $k \geq 0$. *Hint.* Writing

$$f(z) = b_0 + b_1 z + b_2 z^2 + \cdots, \quad (1.3)$$

we get $f(0) = b_0$, $f'(0) = b_1$, $f''(0) = 2! b_2$, and in general only the $n = k$ term of $f^{(k)}(z) = \sum_{n \geq k} n(n-1) \cdots (n-k+1) b_n z^{n-k}$ survives at $z = 0$.

Theorem 1.3. Let f be analytic on an open set U . Then f is holomorphic on U (in fact infinitely $\mathbb{C}$ -differentiable). Moreover, if at $z_0 \in U$ we have an expansion

$$f(z) = \sum b_n (z - z_0)^n \quad \text{for } z \text{ near } z_0, \quad (1.4)$$

then

$$b_k = \frac{f^{(k)}(z_0)}{k!} \quad \text{for all } k \geq 0. \quad (1.5)$$

Proof. Put $g(w) = \sum b_n w^n$, a convergent power series near $w = 0$, so that $f(z) = g(z - z_0)$ for z near z_0 . Translating the variable does not affect difference quotients:

$$\frac{f(z+h) - f(z)}{h} = \frac{g((z - z_0) + h) - g(z - z_0)}{h}. \quad (1.6)$$

So near z_0 , f is infinitely $\mathbb{C}$ -differentiable with $f^{(k)}(z) = g^{(k)}(z - z_0)$, and by the exercise $f^{(k)}(z_0) = g^{(k)}(0) = k! b_k$. Since $z_0 \in U$ was arbitrary, the result follows. $\square$

In short:

$$f \text{ analytic on } U \implies f \text{ holomorphic on } U. \quad (1.7)$$

2 Functions vs series

For an open set $U \subseteq \mathbb{C}$, let us write

$$\mathcal{A}(U) := \{f: U \rightarrow \mathbb{C} \mid f \text{ is analytic on } U\} \text{ and also} \quad (2.1)$$

$$\mathcal{H}(U) := \{f: U \rightarrow \mathbb{C} \mid f \text{ is holomorphic on } U, \text{ i.e., complex differentiable at every } z \in U\}. \quad (2.2)$$

Now we can equip $\mathcal{H}(U)$ with the structure of a ring (actually a $\mathbb{C}$ -algebra) in the natural way: if $f, g \in \mathcal{H}(U)$, then $(f+g)(z) := f(z) + g(z)$ and $(fg)(z) := f(z)g(z)$ are again holomorphic (so belong to $\mathcal{H}(U)$) by what we saw earlier in class. We have just seen that $\mathcal{A}(U) \subset \mathcal{H}(U)$ and we can ask the question: does this algebra structure preserve the space $\mathcal{A}(U)$? In other words, given f, g analytic functions on U , is $f \cdot g$ and $f + g$ an analytic function on U ? This is answered in the following lemma.

Proposition 2.1. Let $f, g \in \mathcal{A}(U)$ be analytic functions on an open set U . Then fg and $f + g$ are again analytic.

Proof. Let us just prove the first statement, as the second follows from a similar (and simpler) argument. Let $z_0 \in U$. As f, g are analytic, there are power series $F(T) = \sum_n a_n T^n$ and $G(T) = \sum_n b_n T^n$ and an $s > 0$ so that F, G both have radius of convergence $\geq s$ and such that $f(z) = F(z - z_0)$ and $g(z) = G(z - z_0)$ for $z \in D(z_0, s)$. What we need to argue is that there exists a power series $H(T) := \sum_n c_n T^n$ with radius of convergence $\geq s$ and such that $(fg)(z) = H(z - z_0)$ for $z \in D(z_0, s)$. Set $c_n := \sum_{k=0}^n a_k b_{n-k}$. We claim that

$$H(T) = F(T)G(T) = \sum_{n=0}^{\infty} c_n T^n \quad (2.3)$$

will work. To establish this first observe:

Lemma 2.2. *Suppose $r > 0$ and $f(T) = \sum a_n T^n$ and $g(T) = \sum b_n T^n$ both have radius of convergence at least r . Then*

$$f(T) \cdot g(T) = \sum c_n T^n, \quad c_n = \sum_{k=0}^n a_k b_{n-k}, \quad (2.4)$$

has radius of convergence $\geq r$.

Warning. Equality need not hold even when F and G both have radius s : for $s = 1$, the series of $F(T) = (1 + T)/(1 - T)$ and $G(T) = (1 - T)/(1 + T)$ both have radius 1, while $F(T)G(T) = 1$ has infinite radius.

Proof. Since $f(T)$ and $g(T)$ have radius of convergence at least r , Hadamard's criterion gives

$$\limsup_n |a_n|^{1/n} \leq \frac{1}{r}, \quad \limsup_n |b_n|^{1/n} \leq \frac{1}{r}. \quad (2.5)$$

Pick any $0 < s < r$.

Claim. There is a $C > 0$ such that

$$|a_n| \leq \frac{C}{s^n} \quad \text{and} \quad |b_n| \leq \frac{C}{s^n} \quad \text{for all } n. \quad (2.6)$$

Indeed, since $0 < s < r$ we have $\frac{1}{s} > \frac{1}{r}$, so there are only finitely many n with $|a_n|^{1/n} \geq \frac{1}{s}$ (and likewise for b_n). For all other n we have $|a_n| < s^{-n}$. Choosing $C \geq 1$ larger than $|a_n|s^n$ and $|b_n|s^n$ for these finitely many n , we get the desired bound.

Now,

$$|c_n| \leq \sum_{k=0}^n |a_k| |b_{n-k}| \leq \sum_{k=0}^n \frac{C}{s^k} \cdot \frac{C}{s^{n-k}} = \frac{C^2(n+1)}{s^n}. \quad (2.7)$$

Hence

$$\limsup_n |c_n|^{1/n} \leq \limsup_n \frac{C^{2/n}(n+1)^{1/n}}{s} = \frac{1}{s}, \quad (2.8)$$

because $C^{2/n} \rightarrow 1$ and $(n+1)^{1/n} = e^{\frac{1}{n} \log(n+1)} \rightarrow e^0 = 1$. Therefore the radius of convergence of fg is $\geq s$. As $s < r$ was arbitrary, the radius of convergence of fg is at least r . $\square$

For every $|w| < s$, both $\sum_n |a_n w^n|$ and $\sum_n |b_n w^n|$ converge, so

$$\sum_{n=0}^{\infty} |c_n w^n| \leq \sum_{n=0}^{\infty} \sum_{k=0}^n |a_k w^k| |b_{n-k} w^{n-k}| = \left(\sum_{k=0}^{\infty} |a_k w^k| \right) \left(\sum_{j=0}^{\infty} |b_j w^j| \right) < \infty.$$

Thus $H(T) = \sum_{n=0}^{\infty} c_n T^n$ has radius of convergence at least s . The same absolute convergence lets us rearrange the double sum, giving

$$H(w) = \sum_{n=0}^{\infty} \sum_{k=0}^n a_k b_{n-k} w^n = \left(\sum_{k=0}^{\infty} a_k w^k \right) \left(\sum_{j=0}^{\infty} b_j w^j \right) = F(w)G(w).$$

Taking $w = z - z_0$ shows that $(fg)(z) = H(z - z_0)$ on $D(z_0, s)$. Since z_0 was arbitrary, fg is analytic on U . □

3 Uniqueness of power series expansions

Question. Suppose f is analytic on $U \ni z_0$. Can we have two different power series expansions $\sum a_n T^n$ and $\sum b_n T^n$ of f near z_0 ?

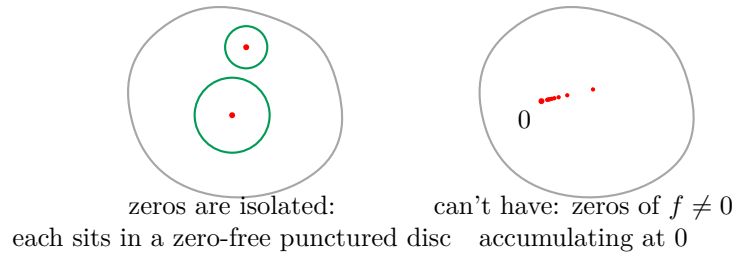
Answer. No! By (1.5), both a_k and b_k equal $f^{(k)}(z_0)/k!$, a number determined by f alone. The following proposition gives a much stronger statement: a convergent power series is already determined by its values on any set of points accumulating at the centre.

Proposition 3.1.

- ① (Isolated zeros.) *Let $f(T) \in \mathbb{C}[[T]]$ have positive radius of convergence r . Assume $f(T) \neq 0$ (i.e. f is not the power series $0 + 0 \cdot T + 0 \cdot T^2 + \dots$), but $f(0) = 0$. Then there exists $s > 0$ such that*

$$f(z) \neq 0 \quad \text{for all } z \in D(0, s), \quad z \neq 0. \quad (3.1)$$

- ② (Identity principle.) *Let $f(T), g(T) \in \mathbb{C}[[T]]$ have positive radii of convergence, and suppose $f(z) = g(z)$ for all z in an infinite subset of their common disc of convergence having 0 as a point of accumulation. Then in fact $f(T) = g(T)$ as formal power series.*



Proof. ① $\Rightarrow$ ②. Let $h(T) = f(T) - g(T)$. It is convergent wherever f and g are, so it has positive radius of convergence, and $h(z) = f(z) - g(z)$ there. Now, as $h(z)$ is continuous, let $\{z_n\}$ be a

sequence of distinct points of the given subset, all nonzero, with $z_n \rightarrow 0$ (possible since 0 is a point of accumulation). Then

$$f(z_n) = g(z_n) \implies h(z_n) = 0 \implies h(0) = \lim_{n \rightarrow \infty} h(z_n) = 0. \quad (3.2)$$

But if $h(T) \neq 0$, then ① provides a punctured disc $D(0, s) \setminus \{0\}$ on which h has no zeros, whereas z_n lies in it for all large n . This contradicts the “isolatedness” of zeros. Hence $h(T) = 0$, i.e. $f(T) = g(T)$.

Proof of ①. Since $f(T) \neq 0$, some coefficient is nonzero; let a_m be the first one. As $f(0) = a_0 = 0$, we have $m \geq 1$, and

$$\begin{aligned} f(T) &= a_m T^m + \text{higher powers of } T \\ &= a_m T^m (1 + \underbrace{b_1 T + b_2 T^2 + \cdots}_{h(T)}) = a_m T^m (1 + h(T)), \quad b_k := \frac{a_{m+k}}{a_m}. \end{aligned}$$

Since $f(T)$ has positive radius of convergence, so does $h(T)$: it is obtained from f by dropping finitely many terms, shifting the powers, and scaling, none of which changes the radius. Moreover, for $z \in D(0, r)$ we have $f(z) = a_m z^m (1 + h(z))$ as functions.

Now $h(z)$ is continuous and $h(0) = 0$. Choose a small neighbourhood $D(0, s)$ of 0, with $s \leq r$, such that $|h(z)| < 1$ there. Then in this neighbourhood,

$$f(z) = a_m z^m (1 + h(z)), \quad a_m \neq 0, \quad 1 + h(z) \neq 0, \quad (3.3)$$

and this means that any zero of f in this neighbourhood is at $z = 0$. □

Remark. The integer $m \geq 1$ in the proof is called the *order* of the zero of f at 0: near 0, f behaves like $a_m z^m$ times a function that does not vanish.