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1 Recollections

To a formal power series $f(T) = \sum a_n T^n \in \mathbb{C}[[T]]$ we have assigned an extended nonnegative number $r \in [0, \infty]$, the *radius of convergence*, such that

$$f(z) = \sum a_n z^n \quad (1.1)$$

converges absolutely at every point of $D(0, r)$, converges uniformly on every closed disc $\overline{D(0, s)}$ with $s < r$, and diverges for $|z| > r$. Hadamard's criterion computes it from the coefficients:

$$\limsup_n |a_n|^{1/n} = \frac{1}{r}. \quad (1.2)$$

Example 1.1. Consider $\sum n! z^n$. By Stirling, $n! = n^n e^{-n} u_n$ with $\lim_{n \rightarrow \infty} u_n^{1/n} = 1$, so

$$\limsup_n |n!|^{1/n} = \limsup_n n e^{-1} u_n^{1/n} = \infty. \quad (1.3)$$

Hence the radius of convergence is $\frac{1}{\infty} = 0$: the series converges only at $z = 0$.

Stirling's formula

A useful estimate to keep in mind is the following:

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \quad \text{as } n \rightarrow \infty, \quad (1.4)$$

where $a_n \sim b_n$ means $a_n/b_n \rightarrow 1$ as $n \rightarrow \infty$. So $n!$ grows like n^n , damped by e^{-n} and corrected by the comparatively tiny factor $\sqrt{2\pi n}$. In Example 1.1 that factor (times something tending to 1) is u_n .

2 The formal derivative has the same radius of convergence

Suppose $f(T) = \sum_{n=0}^{\infty} a_n T^n$ has radius of convergence r . Its *formal derivative* is

$$f'(T) = \sum_{n \geq 1} n a_n T^{n-1}. \quad (2.1)$$

Proposition 2.1. *The formal derivative $\sum_{n \geq 1} n a_n z^{n-1}$ has radius of convergence r as well.*

Proof. First consider the formal series $Tf'(T) = \sum_{n \geq 1} n a_n T^n$. Since $n^{1/n} = e^{\frac{1}{n} \log n} \rightarrow e^0 = 1$ as $n \rightarrow \infty$, Hadamard's criterion gives

$$\limsup_{n \rightarrow \infty} n^{1/n} |a_n|^{1/n} = \limsup_{n \rightarrow \infty} |a_n|^{1/n} = \frac{1}{r}. \quad (2.2)$$

Thus $Tf'(T)$ has radius r . Removing the initial zero coefficient and shifting the powers down by one does not change a power series's radius of convergence, so $f'(T)$ also has radius r . $\square$

3 A convergent power series is $\mathbb{C}$ -differentiable

Let $f(z) = \sum a_n z^n$ be a power series with radius of convergence r and regard it as a function on $D(0, r)$. *Question.* What is the relation between $f'(z)$ — the derivative we discussed earlier — and

$$\sum_{n \geq 1} n a_n z^{n-1}? \quad (3.1)$$

To address this, we must first ask what kind of function $f(z)$ is. On every closed disc $\overline{D(0, s)}$ with $s < r$, it is a uniform limit of $\mathbb{C}$ -differentiable functions, namely the partial sums

$$S_k = \sum_{n=0}^k a_n z^n \rightarrow f. \quad (3.2)$$

Warning. A uniform limit of differentiable functions need not be differentiable in the real-variable setting. For example, the functions

$$f_n(x) = \sqrt{x^2 + \frac{1}{n}} \quad (x \in \mathbb{R}) \quad (3.3)$$

are differentiable (indeed smooth) on $\mathbb{R}$, and they converge uniformly to $|x|$, since

$$0 \leq \sqrt{x^2 + \frac{1}{n}} - |x| = \frac{1/n}{\sqrt{x^2 + \frac{1}{n}} + |x|} \leq \frac{1}{\sqrt{n}}. \quad (3.4)$$

Yet the limit $|x|$ is not differentiable at $x = 0$. (If you have seen it before, then the Weierstrass approximation theorem states that every continuous function on $[a, b]$ — even a nowhere-differentiable one — is a uniform limit of polynomials.)

Theorem 3.1. The function $f(z) = \sum a_n z^n$ is $\mathbb{C}$ -differentiable on its region of convergence $D(0, r)$, and in fact it has derivative given by

$$f'(z) = \sum_{n \geq 1} n a_n z^{n-1} \quad \text{for all } z \in D(0, r). \quad (3.5)$$

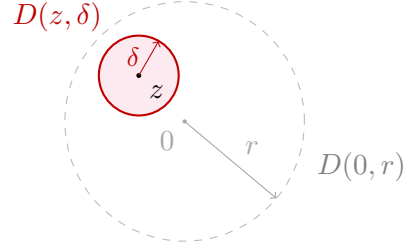
In other words, a power series may be differentiated term by term inside its disc of convergence.

Proof. Remember that f is $\mathbb{C}$ -differentiable at a point $z \in \mathbb{C}$ if there exist a function $\varphi(h)$, defined for h near 0, and a complex number α such that

- $f(z+h) - f(z) - \alpha h = \varphi(h) \cdot h$, and
- $\lim_{h \rightarrow 0} \varphi(h) = 0$.

In that case $f'(z) = \alpha$.

Given $z \in D(0, r)$, choose $\delta > 0$ with $|z| + \delta < r$, so that $D(z, \delta) \subset D(0, r)$.



For $|h| < \delta$ the point $z+h$ lies in $D(z, \delta) \subset D(0, r)$, and by the binomial theorem

$$\begin{aligned} f(z+h) &= \sum_n a_n (z+h)^n = \sum_n a_n \left(z^n + \binom{n}{1} z^{n-1} h + \cdots + h^n \right) \\ &= \sum_n a_n (z^n + n z^{n-1} h + h^2 P_n(z, h)), \end{aligned}$$

where

$$P_n(z, h) := \sum_{k=2}^n \binom{n}{k} z^{n-k} h^{k-2} \quad (\text{and } P_0 = P_1 = 0). \quad (3.6)$$

Why does $\sum a_n P_n(z, h)$ converge? By the definition of P_n , each term satisfies

$$h^2 a_n P_n(z, h) = a_n (z+h)^n - a_n z^n - n a_n z^{n-1} h. \quad (3.7)$$

The three series on the right converge: $\sum a_n (z+h)^n$ and $\sum a_n z^n$ because $z, z+h \in D(0, r)$, and $\sum n a_n z^{n-1}$ by Proposition 2.1. A term-by-term combination of convergent series converges, so $\sum h^2 a_n P_n(z, h)$ converges. For $h \neq 0$ we may divide by h^2 , and so $\sum a_n P_n(z, h)$ converges.¹ Hence

$$f(z+h) - f(z) = h \sum_n n a_n z^{n-1} + h^2 \sum_n a_n P_n(z, h). \quad (3.8)$$

So we set

$$\alpha := \sum_n n a_n z^{n-1}, \quad \varphi(h) := h \sum_n a_n P_n(z, h), \quad (3.9)$$

which exist for $h \in D(0, \delta)$. By construction, $f(z+h) - f(z) - \alpha h = \varphi(h) \cdot h$.

¹When $h = 0$ we cannot divide by h^2 , but then $P_n(z, 0) = \binom{n}{2} z^{n-2}$. So $\sum a_n P_n(z, 0) = \sum \binom{n}{2} a_n z^{n-2}$ is half the second formal derivative of f evaluated at z . It converges by Proposition 2.1 applied twice. (The value at $h = 0$ plays no role in the limit below, since $\varphi(0) = 0$ in any case.)

Check that $\lim_{h \rightarrow 0} \varphi(h) = 0$. First,

$$|\varphi(h)| \leq |h| \sum_n |a_n| |P_n(z, h)|. \quad (3.10)$$

Since $|h| < \delta$ we then have

$$|P_n(z, h)| \leq \sum_{k=2}^n \binom{n}{k} |z|^{n-k} |h|^{k-2} \leq \sum_{k=2}^n \binom{n}{k} |z|^{n-k} \delta^{k-2} = P_n(|z|, \delta). \quad (3.11)$$

Therefore

$$|\varphi(h)| \leq |h| \sum_n |a_n| P_n(|z|, \delta). \quad (3.12)$$

The sum on the right is a *fixed* number that does not depend on h . It is finite because the binomial expansion gives $\delta^2 P_n(|z|, \delta) = (|z| + \delta)^n - |z|^n - n|z|^{n-1}\delta$. So (since $\delta > 0$)

$$\sum_n |a_n| P_n(|z|, \delta) = \frac{1}{\delta^2} \left(\sum_n |a_n| (|z| + \delta)^n - \sum_n |a_n| |z|^n - \delta \sum_n n |a_n| |z|^{n-1} \right), \quad (3.13)$$

and all three series on the right converge because $|z|$ and $|z| + \delta$ are less than r (again using Proposition 2.1 for the last one). Letting $h \rightarrow 0$ in (3.12) gives $\varphi(h) \rightarrow 0$. Thus f is $\mathbb{C}$ -differentiable at z , with

$$f'(z) = \alpha = \sum_{n \geq 1} n a_n z^{n-1}. \quad (3.14)$$

Since $z \in D(0, r)$ was arbitrary, f is $\mathbb{C}$ -differentiable on all of $D(0, r)$ and its derivative is obtained by differentiating term by term. $\square$

Corollary 3.2. *The function $f(z) = \sum a_n z^n$ is infinitely $\mathbb{C}$ -differentiable on $D(0, r)$. For instance,*

$$f''(z) = \sum_{n \geq 2} n(n-1) a_n z^{n-2}, \quad (3.15)$$

and in general, for every $k \geq 0$,

$$f^{(k)}(z) = \sum_{n \geq k} n(n-1) \cdots (n-k+1) a_n z^{n-k} \quad \text{for all } z \in D(0, r). \quad (3.16)$$

Proof. By Proposition 2.1, the series $f'(z) = \sum_{n \geq 1} n a_n z^{n-1}$ again has radius of convergence r . So the same analysis, i.e. Theorem 3.1 applied to f' , shows that f' is $\mathbb{C}$ -differentiable on $D(0, r)$ with derivative $\sum_{n \geq 2} n(n-1) a_n z^{n-2}$. Repeating this argument (induction on k) gives the formula for $f^{(k)}$. $\square$

4 Analytic functions

Definition 4.1 (Analytic). Let $U \subseteq \mathbb{C}$ be an open set. A function $f: U \rightarrow \mathbb{C}$ is said to be *analytic at $z_0 \in U$* if there exist $\rho > 0$ and a power series $\sum a_n T^n$, with radius of convergence at least ρ , such that $D(z_0, \rho) \subseteq U$ and

$$f(z) = \sum a_n (z - z_0)^n \quad \text{for all } |z - z_0| < \rho, \quad \text{i.e., for all } z \in D(z_0, \rho). \quad (4.1)$$

If f is analytic at each point $z_0 \in U$, we say f is *analytic on U* .

Theorem 4.2. *Let $f(T) = \sum a_n T^n$ have radius of convergence r . Then the function $f(z) = \sum a_n z^n$ that it defines is analytic on all of $D(0, r)$.*

Proof. Pick $z_0 \in D(0, r)$ and choose $s > 0$ such that $|z_0| + s < r$. We seek a power series expansion around z_0 of the form

$$f(z) = \sum b_k (z - z_0)^k \quad \text{for } z \in D(z_0, s), \quad (4.2)$$

whose radius of convergence is at least s . For such z ,

$$z = z_0 + (z - z_0), \quad |z| \leq |z_0| + |z - z_0| < r, \quad (4.3)$$

so $z \in D(0, r)$ and $f(z)$ makes sense. Now expand

$$z^n = \sum_{k=0}^n \binom{n}{k} z_0^{n-k} (z - z_0)^k, \quad (4.4)$$

so that

$$f(z) = \sum_n a_n z^n = \sum_n a_n \left(\sum_{k=0}^n \binom{n}{k} z_0^{n-k} (z - z_0)^k \right). \quad (4.5)$$

For each $k \geq 0$, the series

$$b_k := \sum_{n=k}^{\infty} \binom{n}{k} z_0^{n-k} a_n \quad (4.6)$$

converges absolutely: up to the factor $1/k!$, it is the formally differentiated series of order k evaluated at z_0 , and that series still has radius r by Proposition 2.1. The resulting double series is absolutely convergent because

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^n |a_n| \binom{n}{k} |z_0|^{n-k} |z - z_0|^k \\ = \sum_{n=0}^{\infty} |a_n| (|z_0| + |z - z_0|)^n < \infty, \end{aligned}$$

where the last series converges since $|z_0| + |z - z_0| < |z_0| + s < r$.

To rearrange the double series we use the following standard fact.

Rearranging absolutely convergent double series

Let $c_{n,k} \in \mathbb{C}$ for $0 \leq k \leq n$, and suppose $\sum_{n=0}^{\infty} \sum_{k=0}^n |c_{n,k}| < \infty$. Then for each k the series $\sum_{n \geq k} c_{n,k}$ converges absolutely, and

$$\sum_{n=0}^{\infty} \sum_{k=0}^n c_{n,k} = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} c_{n,k}. \quad (4.7)$$

Indeed, write S for the (finite) sum of all $|c_{n,k}|$. Given $\varepsilon > 0$, choose N so large that the terms with $n > N$ contribute less than ε to S . Both iterated sums differ from the finite sum $\sum_{0 \leq k \leq n \leq N} c_{n,k}$ by at most the sum of $|c_{n,k}|$ over the pairs with $n > N$, hence by less than ε . So the two iterated sums agree. In short, an absolutely convergent series can be summed in any order.

We apply this with $c_{n,k} = a_n \binom{n}{k} z_0^{n-k} (z - z_0)^k$. Summing first over n for fixed k gives $b_k (z - z_0)^k$, so we obtain

$$f(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k, \quad (4.8)$$

valid for every $z \in D(z_0, s)$.

Since (4.8) holds for every $z \in D(z_0, s)$, the series $\sum b_k T^k$ converges at every T with $|T| < s$ (take $T = z - z_0$). Hence its radius of convergence is at least s : if it were some $R < s$, the series would diverge at any T with $R < |T| < s$. Together with $f(z) = \sum b_k (z - z_0)^k$ on $D(z_0, s) \subseteq D(0, r)$, this shows that f is analytic at z_0 (take $\rho = s$ in Definition 4.1). Since $z_0 \in D(0, r)$ was arbitrary, f is analytic on $D(0, r)$. $\square$

Example 4.3. Let

$$f(z) = \frac{z^2}{z+2}. \quad (4.9)$$

Near 0:

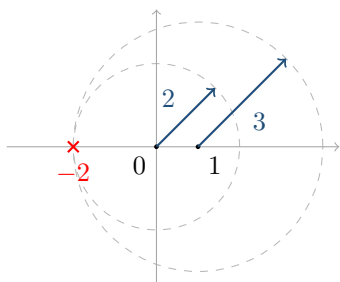
$$\frac{z^2}{z+2} = z^2 \left(\frac{1}{\frac{z}{2} + 1} \right) \cdot \frac{1}{2} = \frac{1}{2} z^2 \left(1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \cdots \right), \quad (4.10)$$

which has radius of convergence 2 — the distance from 0 to the pole at $z = -2$.

Near $z = 1$: writing $z = 1 + (z - 1)$,

$$f(z) = \frac{z^2}{z+2} = \frac{(1 + (z - 1))^2}{1 + (z - 1) + 2} = \frac{1}{3} + \frac{5}{9}(z - 1) + \cdots, \quad (4.11)$$

and this expansion has radius of convergence 3, again the distance from the centre 1 to the pole at -2 .



Remark. In both cases the radius of convergence is exactly the distance from the centre of expansion to the nearest point where f fails to be defined. This is not an accident, and we will return to it.

Exercise 4.4. Expand $f(z) = \frac{z^2}{z+2}$ around $z_0 = -1$ and check that the radius of convergence you obtain from Hadamard's criterion equals $|-1 - (-2)| = 1$.