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1 What does it mean to plug a complex number into a formal power series?

Let $f \in \mathbb{C}[[T]]$ be a formal power series,

$$f = \sum_{n \geq 0} a_n T^n, \quad a_n \in \mathbb{C}. \quad (1.1)$$

The symbol T is an indeterminate: so far nothing has been substituted for it. The question driving this lecture is: *what sense does it make to “plug in” a number $z \in \mathbb{C}$?*

The answer goes through series of functions. Let S be a set and let $f_n: S \rightarrow \mathbb{C}$ be functions on S ; we consider

$$\sum_n f_n. \quad (1.2)$$

The case we actually care about is

$$f_n(z) = a_n z^n, \quad S = D(0, r), \quad (1.3)$$

so that the partial sums are the truncations of the power series.

Definition 1.1 (The three modes of convergence). Write $S_k = \sum_{n=0}^k f_n$ for the k -th partial sum. We say that $\sum f_n$

- converges *pointwise* if, for each $z \in S$, the sequence of complex numbers $\{S_k(z)\}$ converges;
- converges *uniformly* if $\{S_k\}$ converges with respect to the sup norm $\|\cdot\|_S$;
- converges *absolutely* if, setting $V_k(z) = \sum_{n=0}^k |f_n(z)|$, the sequence $\{V_k(z)\}$ converges for each $z \in S$.

What we care about is the implication

$$\sum |a_n| r^n < \infty \implies \sum a_n z^n \text{ converges uniformly and absolutely on } \overline{D(0, r)}, \quad (1.4)$$

which is an instance of the comparison test that we proved last time:

Theorem 1.2 (Comparison test). *Let $c_n \geq 0$ be real numbers with $\sum c_n$ convergent, let $S \subseteq \mathbb{C}$, and let $\{f_n\}$ be functions on S with*

$$\|f_n\|_S \leq c_n \quad \text{for all } n. \quad (1.5)$$

Then $\sum f_n$ converges uniformly and absolutely on S .

Indeed, the application we have in mind is the following:

Corollary 1.3 (Application). *Suppose $\sum |a_n| r^n$ converges for some $r > 0$. Then $\sum a_n z^n$ converges uniformly and absolutely on the closed disc $\overline{D(0, r)}$.*

Proof. Let $S = \overline{D(0, r)} = \{z \in \mathbb{C} : |z| \leq r\}$ and put $f_n(z) = a_n z^n$. Then

$$\|f_n\|_S \leq |a_n| r^n, \quad (1.6)$$

so Theorem 1.2 applies with $c_n = |a_n| r^n$. $\square$

2 The radius of convergence

For a power series $\sum a_n z^n$, let

$$T = \{0\} \cup \{s > 0 : \sum |a_n| s^n \text{ converges}\}, \quad (2.1)$$

and let $r_0 = \sup T \in [0, \infty]$.

Definition 2.1 (Radius of convergence). The extended nonnegative number r_0 above is called the *radius of convergence* (R.O.C.) of $\sum a_n z^n$. Since $0 \in T$, every power series has a well-defined radius of convergence. The extreme cases are:

- $r_0 = \infty$: the series converges absolutely at every $z \in \mathbb{C}$ and uniformly on every closed disc;
- $r_0 = 0$: the series does not converge anywhere except at $z = 0$.

Proposition 2.2. *The series $\sum a_n z^n$ converges absolutely at every point of $D(0, r_0)$ and uniformly on every closed disc $\overline{D(0, s)}$ with $s < r_0$. It diverges for $|z| > r_0$.*

Proof. If $0 \leq s < r_0$, the definition of the supremum gives some $q \in T$ with $s < q$. For $|z| \leq s$,

$$|a_n z^n| \leq |a_n| s^n \leq |a_n| q^n, \quad (2.2)$$

so the comparison test gives uniform and absolute convergence on $\overline{D(0, s)}$.

Now suppose $|z| > r_0$ and $\sum a_n z^n$ converges. Its terms are bounded, so $|a_n z^n| \leq M$ for some $M > 0$. Choose q with $r_0 < q < |z|$. Then

$$\sum_{n \geq 0} |a_n| q^n \leq M \sum_{n \geq 0} \left(\frac{q}{|z|} \right)^n < \infty, \quad (2.3)$$

which says $q \in T$, contradicting $q > \sup T = r_0$. $\square$

Warning. When $0 < r_0 < \infty$, nothing above says what happens on the circle $|z| = r_0$ itself. The behaviour there genuinely depends on the series: it may converge at every boundary point, at none, or at some but not others.

3 Interlude: $\limsup$

Hadamard's theorem, stated below, computes r_0 directly from the coefficients, but it is phrased using $\limsup$. So: what does $\limsup$ mean?

Definition 3.1 (Accumulation point). Let $(t_n)_{n \geq 0}$ be an infinite sequence of nonnegative real numbers. An *accumulation point* of $(t_n)_{n \geq 0}$ is a point $t \in \mathbb{R}$ such that every interval around t contains t_n for infinitely many indices n .

Fact. Every bounded infinite sequence in $\mathbb{R}$ has a convergent subsequence and hence an accumulation point.

Definition 3.2 ($\limsup$). Suppose $(t_n)_{n \geq 0}$ is a bounded infinite sequence of nonnegative real numbers, and let S be the set of all its accumulation points. Then

$$\limsup_n t_n = \text{l.u.b.}(S). \quad (3.1)$$

A practical way of viewing $\limsup$ is that it is the number $\lambda \in \mathbb{R}$ characterised by: for every $\epsilon > 0$,

- (A) there exist only *finitely many* n with $t_n \geq \lambda + \epsilon$;
- (B) there exist *infinitely many* n with $t_n \geq \lambda - \epsilon$.

Definition 3.3 (Convention). If $(t_n)_{n \geq 0}$ is an unbounded infinite sequence of nonnegative real numbers, we set $\limsup_n t_n = \infty$.

Exercise 3.4. Let $(t_n)_{n \geq 0}$ and $(s_n)_{n \geq 0}$ be bounded infinite sequences of nonnegative real numbers and set $t = \limsup_n t_n$, $s = \limsup_n s_n$. Verify:

- $\limsup_n (t_n + s_n) \leq t + s$;
- $\limsup_n (t_n s_n) \leq t \cdot s$;
- if $\lim_{n \rightarrow \infty} t_n$ exists, then $\limsup_n t_n = \lim_{n \rightarrow \infty} t_n$.

4 Hadamard's criterion

Theorem 4.1 (Hadamard). Let r_0 be the radius of convergence of $\sum a_n z^n$. Then

$$\frac{1}{r_0} = \limsup_n |a_n|^{1/n}. \quad (4.1)$$

Here we use the conventions $1/0 = \infty$ and $1/\infty = 0$.

Proof. Let $t = \limsup_{n \rightarrow \infty} |a_n|^{1/n}$ and assume $0 < t < \infty$ (the extreme cases are treated in the remark below). We must show that $\sum a_n z^n$ converges for $|z| < 1/t$ and diverges for $|z| > 1/t$; this identifies the radius of convergence as $r_0 = 1/t$.

Convergence. By characterisation (A) above, for any $\epsilon > 0$ there are only finitely many n with

$$|a_n|^{1/n} \geq t + \epsilon, \quad \text{i.e.} \quad |a_n| \geq (t + \epsilon)^n. \quad (4.2)$$

Equivalently,

$$|a_n| < (t + \epsilon)^n \quad \text{for almost all } n. \quad (4.3)$$

Let $z \in \mathbb{C}$ with $|z| < 1/t$. Choose $\epsilon > 0$ small enough that

$$|z|(t + \epsilon) < 1, \quad \text{i.e.} \quad |z| < \frac{1}{t + \epsilon}. \quad (4.4)$$

Then, using (4.3) for all but finitely many terms (the finitely many exceptions contribute a constant c),

$$\sum |a_n| |z|^n \leq c + \sum ((t + \epsilon)|z|)^n, \quad (4.5)$$

and the latter is a convergent geometric series since $(t + \epsilon)|z| < 1$. By comparison, $\sum |a_n| |z|^n$ converges.

Divergence. Assume $|z| > 1/t$ and choose ϵ with $0 < \epsilon < t$ such that $|z| = \frac{1}{t - \epsilon}$. For this $\epsilon > 0$, characterisation (B) gives infinitely many n with

$$|a_n|^{1/n} \geq t - \epsilon, \quad \text{hence} \quad |a_n| \geq (t - \epsilon)^n. \quad (4.6)$$

Therefore, for infinitely many n ,

$$|a_n z^n| \geq (t - \epsilon)^n \cdot \frac{1}{(t - \epsilon)^n} = 1. \quad (4.7)$$

In particular, the terms $a_n z^n$ do not tend to 0, so $\sum a_n z^n$ diverges. $\square$

Remark. If the limsup in Hadamard's criterion is zero, then the series converges everywhere, so $r = \infty$. If the limsup is infinite, then $r = 0$.

Example 4.2. Consider

$$\sum_{n \geq 0} \frac{z^n}{n!}, \quad a_n = \frac{1}{n!}. \quad (4.8)$$

To apply Hadamard, look at $\limsup_n \left(\frac{1}{n!} \right)^{1/n}$. By Stirling's formula, $n! = n^n e^{-n} u_n$ with

$\lim_{n \rightarrow \infty} u_n^{1/n} = 1$. Plugging this in gives

$$\limsup_n \left(\frac{1}{n!} \right)^{1/n} = 0, \quad (4.9)$$

so the radius of convergence is ∞ : the exponential series converges on all of $\mathbb{C}$, as we saw last lecture.

Next time

Let $f(z) = \sum a_n z^n$ have radius of convergence r . What is the radius of convergence of the formally differentiated series

$$\sum_{n \geq 1} n a_n z^{n-1} ? \quad (4.10)$$

It turns out to be r again — and this is the first step towards showing that a convergent power series defines a $\mathbb{C}$ -differentiable function.