

Contents

1	Recollections from last time	1
2	Uniform limits of continuous functions	1
3	Uniformly Cauchy sequences	2
4	Series of functions	3
5	Radius of convergence	5

1 Recollections from last time

- Convergence of a sequence (z_n) , $z_n \in \mathbb{C}$, and of series $\sum a_n$, $\sum |a_n|$.
- For $S \subseteq \mathbb{C}$ and $f : S \rightarrow \mathbb{C}$ bounded, the sup norm

$$\|f\|_S = \sup_{z \in S} |f(z)|, \quad (1.1)$$

Given a sequence (f_n) of functions on S , what does $f_n \rightarrow f$ mean?

Definition 1.1. Let $f_n, f : S \rightarrow \mathbb{C}$.

- We say f_n *converges pointwise* to f if $f_n(z) \rightarrow f(z)$ for all $z \in S$.
- We say f_n *converges uniformly* to f if, given $\epsilon > 0$, there exists n_0 such that if $n \geq n_0$, then

$$|f_n(z) - f(z)| < \epsilon \quad \text{for every } z \in S. \quad (1.2)$$

Equivalently, for every $\epsilon > 0$, there exists n_0 such that $\|f_n - f\|_S < \epsilon$ whenever $n \geq n_0$.

Remark. The difference between the two notions of convergence is the order of the quantifiers: for pointwise convergence the index n_0 may depend on the point $z \in S$, whereas for uniform convergence a single n_0 must work for every $z \in S$ at once.

2 Uniform limits of continuous functions

Theorem 2.1. Suppose (f_n) is a sequence of continuous functions on S and $f_n \rightarrow f$ uniformly. Then f is also continuous.

Proof. Fix $z \in S$. We want to show f is continuous at z , i.e.

$$\lim_{w \rightarrow z} f(w) = f(z) : \quad (2.1)$$

given $\epsilon > 0$, there exists $\delta > 0$ such that if $|w - z| < \delta$ then $|f(z) - f(w)| < \epsilon$.

Step 1. Given $\epsilon/3$, by uniformity there exists n_0 such that if $n \geq n_0$ then

$$\|f - f_n\|_S < \frac{\epsilon}{3}. \quad (2.2)$$

Step 2. Since each f_n is continuous, pick some $n \geq n_0$; continuity of f_n at z means there exists $\delta > 0$ such that if $|z - w| < \delta$ then

$$|f_n(z) - f_n(w)| < \frac{\epsilon}{3}. \quad (2.3)$$

Step 3. Now for that same δ , we know that whenever $|z - w| < \delta$,

$$\begin{aligned} |f(z) - f(w)| &\leq \underbrace{|f(z) - f_n(z)|}_{\leq \epsilon/3 \text{ (uniformity)}} + \underbrace{|f_n(z) - f_n(w)|}_{< \epsilon/3 \text{ (} f_n \text{ continuous)}} + \underbrace{|f_n(w) - f(w)|}_{\leq \epsilon/3 \text{ (uniformity)}} \\ &< \epsilon. \end{aligned}$$

Hence f is continuous at z , and since $z \in S$ was arbitrary, f is continuous on S . □

3 Uniformly Cauchy sequences

Another useful notion is the following:

Definition 3.1. Let (f_n) be a sequence of functions on S . We say that (f_n) is *uniformly Cauchy on S* if, given $\epsilon > 0$, there exists n_0 such that if $n, m \geq n_0$, then

$$\|f_n - f_m\|_S < \epsilon. \quad (3.1)$$

Lemma 3.2. Let (f_n) be uniformly Cauchy on S . Then

- (a) $f(z) = \lim_{n \rightarrow \infty} f_n(z)$ exists for each $z \in S$;
- (b) $f_n \rightarrow f$ uniformly.

Proof. (a) Use that, for fixed $z \in S$,

$$|f_n(z) - f_m(z)| \leq \|f_n - f_m\|_S. \quad (3.2)$$

So $(f_n(z))$ is a Cauchy sequence of complex numbers, and hence converges. Call the limit $f(z)$.

(b) Recall that $f_n \rightarrow f$ uniformly means: given $\epsilon > 0$ there exists n_0 such that if $n \geq n_0$ then $\|f_n - f\|_S < \epsilon$.

Since (f_n) is uniformly Cauchy, there exists $N = N(\epsilon)$ such that

$$\|f_n - f_m\|_S < \frac{\epsilon}{2} \quad \text{if } n, m \geq N. \quad (3.3)$$

Fix $n \geq N$ and $z \in S$. For every $m \geq N$ we have

$$|f_n(z) - f_m(z)| \leq \|f_n - f_m\|_S < \frac{\epsilon}{2}. \quad (3.4)$$

Letting $m \rightarrow \infty$ and using part (a), we obtain

$$|f_n(z) - f(z)| \leq \frac{\epsilon}{2}. \quad (3.5)$$

This holds for every $z \in S$, so $\|f_n - f\|_S \leq \epsilon/2 < \epsilon$ whenever $n \geq N$. Thus $f_n \rightarrow f$ uniformly. $\square$

4 Series of functions

Applications

What we are interested in is convergence of power series of the form

$$\sum a_n z^n, \quad a_n \in \mathbb{C}, \quad (4.1)$$

whose partial sums

$$S_n = \sum_{j=0}^n a_j z^j \quad (4.2)$$

are functions on S ; we are interested in when they converge to a function f on S .

Definition 4.1 (Convergence of a series of functions). Let $(f_j)_{j \geq 1}$ be a sequence of functions on S , and put

$$S_d = \sum_{j=1}^d f_j. \quad (4.3)$$

We say that $\sum f_j$ converges *pointwise* (respectively, *uniformly*) to f if $S_d \rightarrow f$ pointwise (respectively, uniformly). We say that $\sum f_j$ converges *absolutely on S* if

$$\sum_{j=1}^{\infty} |f_j(z)| < \infty \quad \text{for every } z \in S. \quad (4.4)$$

If the series of functions $\sum |f_j|$ converges uniformly, we say that $\sum f_j$ converges *uniformly absolutely* on S .

Theorem 4.2 (Comparison test for series of functions). Let (c_n) be a sequence of nonnegative real numbers such that $\sum_{n=1}^{\infty} c_n$ converges. Let (f_n) be a sequence of functions on S satisfying

$$\|f_n\|_S \leq c_n, \quad (4.5)$$

Then $\sum f_n$ and $\sum |f_n|$ converge uniformly on S . In particular, $\sum f_n$ converges absolutely on S . If, moreover, the f_n are continuous, then the limit is continuous.

Proof. For $m < n$ the partial sums satisfy

$$\|S_n - S_m\|_S = \left\| \sum_{j=m+1}^n f_j \right\|_S \leq \sum_{j=m+1}^n \|f_j\|_S \leq \sum_{j=m+1}^n c_j, \quad (4.6)$$

which is small for m, n large because $\sum c_n$ converges (its partial sums are Cauchy). So (S_n) is uniformly Cauchy, and Lemma 3.2 gives a limit f with $S_n \rightarrow f$ uniformly. The same estimate applied to $\sum |f_j|$, using $\| |f_j| \|_S = \|f_j\|_S \leq c_j$, gives uniform convergence of $\sum |f_j|$. Hence $\sum f_j$ converges absolutely at every point of S . Continuity of the limit follows from Theorem 2.1. $\square$

Theorem 4.3. Let (a_n) be a sequence of complex numbers and let $r > 0$. If

$$\sum |a_n| r^n \text{ converges,} \quad (4.7)$$

then $\sum a_n z^n$ converges absolutely and uniformly on

$$D(0, r) = \{z \in \mathbb{C} : |z| < r\} \quad (4.8)$$

to a continuous function.

Proof. Put $f_n(z) = a_n z^n$ on $D(0, r)$. Then

$$\|f_n\|_{D(0, r)} \leq |a_n| r^n. \quad (4.9)$$

Use the comparison test (Theorem 4.2) with $c_n = |a_n| r^n$: it follows that $\sum a_n z^n$ converges uniformly and absolutely on $D(0, r)$, and the limit is continuous since each f_n is. $\square$

Example 4.4. Consider

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad \text{so } a_n z^n = \frac{z^n}{n!}, \quad a_n = \frac{1}{n!}. \quad (4.10)$$

Claim. The exponential series $\sum a_n r^n$ converges for every $r > 0$.

Proof: compare to a geometric series. Fix a positive integer $N \geq 2r$. For $n > N$,

$$\frac{r^n}{n!} = \frac{r^N}{N!} \cdot \frac{r^{n-N}}{(N+1)(N+2) \cdots n} \leq \frac{r^N}{N!} \left(\frac{r}{N}\right)^{n-N} \leq \frac{r^N}{N!} \left(\frac{1}{2}\right)^{n-N}, \quad (4.11)$$

and $\sum (1/2)^{n-N}$ converges. So by the comparison test for series of complex numbers (Lecture 6) the series converges.

Therefore, by Theorem 4.3,

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad (4.12)$$

exists for all $z \in \mathbb{C}$, and defines a continuous function on $\mathbb{C}$.

5 Radius of convergence

Let $f(T) \in \mathbb{C}[[T]]$, say

$$f(T) = \sum a_n T^n. \tag{5.1}$$

Theorem 5.1 (Radius of convergence). *Given any $f(T) \in \mathbb{C}[[T]]$, there exists r with $0 \leq r \leq \infty$ such that*

- $\sum a_n z^n$ converges absolutely for $|z| < r$, and uniformly on $D(0, s)$ for every $0 < s < r$;
- $\sum a_n z^n$ diverges for all $|z| > r$.

Remark. Uniform convergence on all of $D(0, r)$ can fail, which is why the statement is made on the discs $D(0, s)$, $s < r$. For instance $\sum z^n$ has $r = 1$, and the convergence is uniform on each $D(0, s)$ with $s < 1$, but not on $D(0, 1)$.