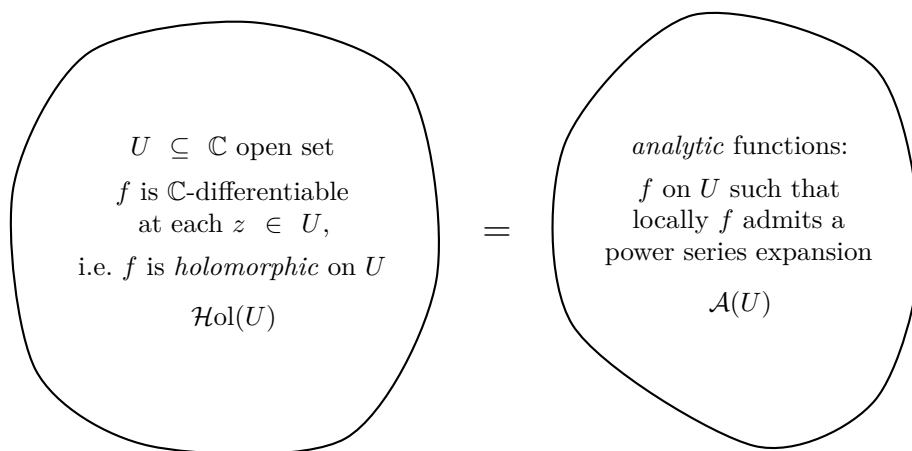


Contents

1	Where we are going: holomorphic versus analytic	1
2	Formal power series	2
3	Sequences and convergence	4
4	Sequences of functions	6

1 Where we are going: holomorphic versus analytic

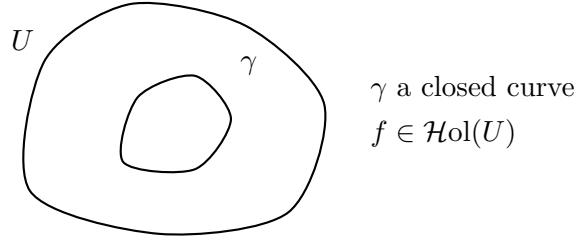
Let $U \subseteq \mathbb{C}$ be an open set. Two classes of functions on U will occupy us for the next several lectures.



The main theorem of this part of the course is that these two classes coincide: $\mathcal{H}\text{ol}(U) = \mathcal{A}(U)$.

Fact (“proved” last time using Green’s theorem). Let $\gamma: [a, b] \rightarrow U$ be a piecewise smooth, simple closed curve whose interior lies in U , and let $f \in \mathcal{H}\text{ol}(U)$. Then

$$\int_{\gamma} f = 0, \quad \text{where} \quad \int_{\gamma} f := \int_a^b f(\gamma(t)) \gamma'(t) dt. \quad (1.1)$$



We will prove this for *analytic* functions directly (i.e. with no Green's theorem), and then, once we know $\mathcal{A}(U) = \mathcal{H}\text{ol}(U)$, it will also hold for $\mathcal{H}\text{ol}(U)$. The next few lectures therefore develop the machinery of power series.

2 Formal power series

Let T be a *formal variable*.

Definition 2.1. $\mathbb{C}[[T]]$ denotes the set of all expressions of the form

$$\sum_{n \geq 0} a_n T^n = a_0 + a_1 T + a_2 T^2 + a_3 T^3 + \cdots, \quad a_n \in \mathbb{C}. \quad (2.1)$$

These are the *formal power series* over $\mathbb{C}$. No convergence is required: a formal power series is just the sequence $(a_0, a_1, a_2, \dots)$ of its coefficients, written suggestively.

What can we do with $\mathbb{C}[[T]]$? Let

$$f = a_0 + a_1 T + a_2 T^2 + \cdots, \quad g = b_0 + b_1 T + b_2 T^2 + \cdots. \quad (2.2)$$

Addition

$$f + g = (a_0 + b_0) + (a_1 + b_1)T + (a_2 + b_2)T^2 + \cdots. \quad (2.3)$$

Multiplication

Multiplying power series:

$$\begin{aligned} f \cdot g &= (a_0 + a_1 T + a_2 T^2 + \cdots)(b_0 + b_1 T + b_2 T^2 + \cdots) \\ &= a_0 b_0 + (a_1 b_0 + a_0 b_1)T + (a_2 b_0 + a_1 b_1 + a_0 b_2)T^2 + \cdots \end{aligned}$$

that is,

$$f \cdot g = \sum_{n \geq 0} \left(\sum_{\substack{j+k=n \\ j,k \geq 0}} a_j b_k \right) T^n. \quad (2.4)$$

Note that each coefficient is a *finite* sum, so this makes sense without any convergence hypothesis.

2.1 Inversion

When does $f \in \mathbb{C}[[T]]$ have a multiplicative inverse? Equivalently, when can we find $g \in \mathbb{C}[[T]]$ such that $f \cdot g = 1$?

Take f fixed and f, g as before. We want

$$a_0 b_0 + (a_1 b_0 + a_0 b_1)T + (a_0 b_2 + a_1 b_1 + a_2 b_0)T^2 + \cdots = 1, \quad (2.5)$$

i.e. $a_0 b_0 = 1$ and every higher coefficient vanishes. To solve the first equation we need $a_0 \neq 0$.

So assume $a_0 \neq 0$, and solve for the b_n recursively:

$$\begin{aligned} a_0 b_0 &= 1 & \implies & b_0 = a_0^{-1}, \\ a_1 b_0 + a_0 b_1 &= 0 & \implies & b_1 = -a_0^{-1}(a_1 b_0), \\ a_0 b_2 + a_1 b_1 + a_2 b_0 &= 0 & \implies & b_2 = -a_0^{-1}(a_1 b_1 + a_2 b_0), \end{aligned}$$

and in general, having solved for $b_0, \dots, b_{n-1}$,

$$b_n = -a_0^{-1} \sum_{j=1}^n a_j b_{n-j}. \quad (2.6)$$

Proposition 2.2. *A power series $f = a_0 + a_1 T + \cdots \in \mathbb{C}[[T]]$ has an inverse if and only if $a_0 \neq 0$.*

Example 2.3. $f(T) = T$ is not invertible in $\mathbb{C}[[T]]$: its constant term is 0.

Remark. If we set

$$\mathbb{C}((T)) = \text{all Laurent series of the form } \sum_{n \geq n_0} a_n T^n \quad (n_0 \in \mathbb{Z}), \quad (2.7)$$

then $\mathbb{C}((T))$ is a *field*, and T does have an inverse in $\mathbb{C}((T))$, namely T^{-1} .

Example 2.4. Let $f = 1 - T$. Solving $(1 - T)(b_0 + b_1 T + b_2 T^2 + \cdots) = 1$ coefficient by coefficient gives

$$b_0 = 1, \quad b_1 - b_0 = 0 \implies b_1 = 1, \quad b_2 - b_1 = 0 \implies b_2 = 1, \quad \dots \quad (2.8)$$

so the inverse $(1 - T)^{-1}$ looks like

$$(1 - T)^{-1} = 1 + T + T^2 + T^3 + \cdots. \quad (2.9)$$

2.2 Composition

A more complicated operation on power series is called *composition*. Given $f(T)$ and $g(T)$, define $(f \circ g)(T)$ as

$$(f \circ g)(T) = a_0 + a_1 g(T) + a_2 g(T)^2 + a_3 g(T)^3 + \cdots. \quad (2.10)$$

This may or may not exist.

Example 2.5. Let $f = 1 + T + T^2 + \cdots$ and $g(T) = 1$. Then

$$f \circ g = 1 + 1 + 1 + \cdots \quad (2.11)$$

does not exist: the constant coefficient is an infinite sum of 1's.

Proposition 2.6. Assume $g(T)$ has no constant term, i.e. $b_0 = 0$. Then $f \circ g$ exists as a power series.

Proof. The expression $\sum_{n \geq 0} a_n g(T)^n$ must be re-expanded as a series in T with well-defined coefficients. Since $b_0 = 0$, the series $g(T)^n$ is divisible by T^n . Hence only the terms with $n \leq m$ contribute to the coefficient of T^m , so that coefficient is a finite sum. $\square$

2.3 Factorization

Finally we state, without proof, the following simple observation.

Proposition 2.7. Every $f(T) \in \mathbb{C}((T))$, $f \neq 0$, can be written as

$$f(T) = c \cdot T^k \cdot u(T), \quad c \in \mathbb{C}, \quad c \neq 0, \quad k \in \mathbb{Z}, \quad (2.12)$$

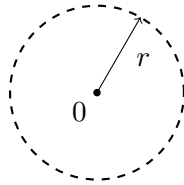
where $u(T) = 1 + a_1 T + a_2 T^2 + \cdots$ is a power series with constant term 1 (hence invertible).

Next goal: assigning a radius to a power series

We want to assign to each $f(T) \in \mathbb{C}[[T]]$ a radius $r \in [0, \infty]$ so that if

$$z \in D(0, r) = \{z \in \mathbb{C} : |z| < r\}, \quad (2.13)$$

then the series $f(z) = \sum_{n \geq 0} a_n z^n$ converges, while if $|z| > r$ it diverges.



The extended nonnegative number r is called the *radius of convergence* of f . To make sense of all this, we first need to recall convergence in $\mathbb{C}$.

Warning. When $r < \infty$, we make no assertion about convergence on the circle $|z| = r$; in fact, almost anything can happen there!

3 Sequences and convergence

We first review some standard notions concerning convergence of series, phrased in the setting of complex numbers.

Definition 3.1. Let $\{z_n\}$ be a sequence of complex numbers.

1. We write

$$w = \lim_{n \rightarrow \infty} z_n \quad (3.1)$$

if, given any $\epsilon > 0$, there exists n_0 such that if $n > n_0$ then

$$|w - z_n| < \epsilon. \quad (3.2)$$

2. We say that $\{z_n\}$ is a *Cauchy sequence* if, given any $\epsilon > 0$, there exists n_0 such that if $m, n > n_0$ then

$$|z_m - z_n| < \epsilon. \quad (3.3)$$

Remark. Write $z_n = x_n + iy_n$. Then

$$|z_n - z_m| = \sqrt{(x_n - x_m)^2 + (y_n - y_m)^2}. \quad (3.4)$$

Since $|x_n - x_m| \leq |z_n - z_m|$ and $|y_n - y_m| \leq |z_n - z_m|$, while $|z_n - z_m| \leq |x_n - x_m| + |y_n - y_m|$, we may use this to argue that

$$\{z_n\} \text{ Cauchy} \iff \{x_n\} \text{ and } \{y_n\} \text{ are Cauchy sequences.} \quad (3.5)$$

Exercise 3.2. Show that $\{z_n\}$ converges $\iff \{z_n\}$ is Cauchy. (Use the same argument as in $\mathbb{R}$ -analysis together with the remark above.)

Next, we turn to the notion of convergence of a series.

Definition 3.3. Let $a_n \in \mathbb{C}$.

1. $\sum_{n=0}^{\infty} a_n$ *converges* if the sequence $\{S_n\}$ of *partial sums* converges, where

$$S_0 = a_0, \quad S_1 = a_0 + a_1, \quad S_2 = a_0 + a_1 + a_2, \quad \dots, \quad S_n = a_0 + \dots + a_n. \quad (3.6)$$

2. $\sum_{n=0}^{\infty} a_n$ is *absolutely convergent* if $\sum_{n=0}^{\infty} |a_n|$ is convergent.

Observe that, just as in real analysis, if $\sum a_n$ is absolutely convergent, then it is convergent. Let

$$S_n := \sum_{i=0}^n a_i \quad \text{and} \quad T_n := \sum_{i=0}^n |a_i|. \quad (3.7)$$

Since $\sum |a_n|$ converges, the sequence $\{T_n\}$ is Cauchy. If $n > m$, then

$$|S_n - S_m| = \left| \sum_{i=m+1}^n a_i \right| \leq \sum_{i=m+1}^n |a_i| = T_n - T_m = |T_n - T_m|. \quad (3.8)$$

Thus $\{S_n\}$ is Cauchy and therefore converges. Hence $\sum a_n$ converges.

Theorem 3.4 (Comparison test). *If $\sum c_n$ is a convergent series of nonnegative real numbers $c_n \geq 0$, and $\sum a_n$ is a series such that $|a_n| \leq c_n$ for every $n \geq 0$, then $\sum a_n$ converges.*

Proof. The partial sums $\sum_{k=0}^n c_k$ are bounded above, since $\sum c_n$ converges. Moreover,

$$\sum_{k=0}^n |a_k| \leq \sum_{k=0}^n c_k. \quad (3.9)$$

Thus the partial sums of $\sum_{n=0}^{\infty} |a_n|$ are bounded above.

From $\mathbb{R}$ -analysis: if $\{b_n\}$ is a sequence of nonnegative numbers such that the partial sums $t_k = \sum_{n=0}^k b_n$ are bounded above, then by the least upper bound property¹ the sequence $\{t_k\}$ converges because it is increasing and bounded above. Applying this to $b_n = |a_n|$ shows $\sum |a_n|$ converges, i.e. $\sum a_n$ converges absolutely, and hence converges. $\square$

4 Sequences of functions

Let $S \subseteq \mathbb{C}$ be nonempty. For a bounded function $h : S \rightarrow \mathbb{C}$, define its *sup norm* by

$$\|h\|_S := \sup_{z \in S} |h(z)|. \quad (4.1)$$

Definition 4.1. Let $f_n, f : S \rightarrow \mathbb{C}$.

- We say f_n *converges pointwise* to f if $f_n(z) \rightarrow f(z)$ for all $z \in S$.
- We say f_n *converges uniformly* to f if, given $\epsilon > 0$, there exists n_0 such that if $n \geq n_0$, then

$$|f_n(z) - f(z)| < \epsilon \quad \text{for every } z \in S. \quad (4.2)$$

Equivalently, for every $\epsilon > 0$, there exists n_0 such that $\|f_n - f\|_S < \epsilon$ whenever $n \geq n_0$.

Definition 4.2. The sequence $\{f_n\}$ is *uniformly Cauchy* if, given $\epsilon > 0$, there is n_0 such that $|f_n(z) - f_m(z)| < \epsilon$ for every $z \in S$ whenever $m, n \geq n_0$. Equivalently, $\|f_n - f_m\|_S < \epsilon$.

Remark. The difference between the two notions of convergence is the order of the quantifiers: for pointwise convergence the index n_0 may depend on the point $z \in S$, whereas for uniform convergence a single n_0 must work for every $z \in S$ at once. We take this up next time.

¹Every nonempty subset of $\mathbb{R}$ that is bounded above has a least upper bound (supremum) in $\mathbb{R}$. More precisely, if $S \subseteq \mathbb{R}$ is nonempty and bounded above, then there exists $s = \sup S \in \mathbb{R}$ such that $x \leq s$ for every $x \in S$, and for every $\epsilon > 0$ there exists $x \in S$ such that $s - \epsilon < x \leq s$.