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1 Completing the proof of the Cauchy–Riemann criterion

Last time we almost finished the proof of the following equivalence. Throughout, $U \subseteq \mathbb{C}$ is open, we identify $\mathbb{C}$ with $\mathbb{R}^2$ via $z = x + iy \leftrightarrow \mathbf{z} = \begin{pmatrix} x \\ y \end{pmatrix}$, and we write

$$f : U \rightarrow \mathbb{C}, \quad f(z) = u(x, y) + iv(x, y), \quad F : U \rightarrow \mathbb{R}^2, \quad F = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}. \quad (1.1)$$

Theorem 1.1 (Cauchy–Riemann criterion).

$$\left\{ \begin{array}{l} F \text{ is totally differentiable at } \mathbf{z} \in U \\ \text{and } J_F = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \text{ satisfies CR} \\ \text{(i.e. } u_x = v_y, u_y = -v_x \text{)} \end{array} \right\} \iff \left\{ \begin{array}{l} f(z) = u(x, y) + iv(x, y) \\ \text{is } \mathbb{C}\text{-differentiable} \\ \text{at } z \in U \subseteq \mathbb{C} \end{array} \right\} \quad (1.2)$$

The implication ($\Leftarrow$) was proved last time: complex differentiability of f at z was shown to force total differentiability of F at $\mathbf{z}$ together with the Cauchy–Riemann equations.

Let us now prove ($\Rightarrow$). What we must show is that there exist a complex number $\xi = a + ib \in \mathbb{C}$ and a function $\sigma(w)$, defined near $w = 0$, such that

1. $f(z + w) - f(z) = \xi w + \sigma(w)w$;
2. $\lim_{w \rightarrow 0} \sigma(w) = 0$.

How do we produce such a $\xi \in \mathbb{C}$? All we know is that $F = \begin{pmatrix} u \\ v \end{pmatrix}$ is totally differentiable at $\mathbf{z}$ and that its Jacobian satisfies the Cauchy–Riemann equations.

1.1 Producing ξ

Since F satisfies CR at $\mathbf{z}$, the derivative matrix has the special shape

$$D_{\mathbf{z}} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad a = u_x = v_y, \quad b = v_x = -u_y, \quad (1.3)$$

for some $a, b \in \mathbb{R}$. This is precisely the matrix of multiplication by the complex number

$$\boxed{\xi = a + ib \in \mathbb{C}.} \quad (1.4)$$

Indeed, if $w = h + ik$ corresponds to $\mathbf{w} = \begin{pmatrix} h \\ k \end{pmatrix}$, then

$$\xi w = (a + ib)(h + ik) = (ah - bk) + i(ak + bh), \quad D_{\mathbf{z}}\mathbf{w} = \begin{pmatrix} ah - bk \\ ak + bh \end{pmatrix}, \quad (1.5)$$

so that $D_{\mathbf{z}}\mathbf{w}$ is exactly the pair of real and imaginary parts of ξw . Consequently the real expression

$$F(\mathbf{z} + \mathbf{w}) - F(\mathbf{z}) - D_{\mathbf{z}}\mathbf{w} = \begin{pmatrix} \varphi_1(h, k) \\ \varphi_2(h, k) \end{pmatrix} \quad (1.6)$$

matches, componentwise, the complex expression

$$f(z + w) - f(z) - \xi w = \varphi_1(h, k) + i \varphi_2(h, k). \quad (1.7)$$

From the definition of total differentiability of F at $\mathbf{z}$ we know that

$$\lim_{\mathbf{w} \rightarrow 0} \frac{\varphi_1(h, k)}{\|\mathbf{w}\|} = \lim_{\mathbf{w} \rightarrow 0} \frac{\varphi_2(h, k)}{\|\mathbf{w}\|} = 0. \quad (1.8)$$

1.2 Producing σ

Define $\sigma(w)$ using the expression above: for $w \neq 0$ set

$$\sigma(w) \cdot w = \varphi_1(h, k) + i \varphi_2(h, k), \quad \text{i.e.} \quad \sigma(w) = \frac{\varphi_1(h, k) + i \varphi_2(h, k)}{w}, \quad (1.9)$$

and set $\sigma(0) = 0$. With this definition, property (1) holds by construction.

Exercise 1.2. Check that $\lim_{w \rightarrow 0} \sigma(w) = 0$, using the two limits recorded above.

Remark. Here is the check. Since $|w| = \sqrt{h^2 + k^2} = \|\mathbf{w}\|$, for $w \neq 0$,

$$|\sigma(w)| = \frac{|\varphi_1(h, k) + i \varphi_2(h, k)|}{|w|} = \frac{\sqrt{\varphi_1(h, k)^2 + \varphi_2(h, k)^2}}{\|\mathbf{w}\|} \leq \frac{|\varphi_1(h, k)|}{\|\mathbf{w}\|} + \frac{|\varphi_2(h, k)|}{\|\mathbf{w}\|} \rightarrow 0. \quad (1.10)$$

Hence $\sigma(w) \rightarrow 0$ as $w \rightarrow 0$, which is (2). Therefore f is complex differentiable at z , with $f'(z) = \xi = u_x + i v_x$.

2 A simple application: locally constant functions

Definition 2.1. We say that $f : U \rightarrow \mathbb{C}$ is *locally constant* if for each $z \in U$ there is a neighbourhood $D(z, \epsilon) \subseteq U$ such that

$$f|_{D(z, \epsilon)} = \text{constant}. \quad (2.1)$$

A locally constant function need not be constant: it is constant on each “piece” of U , but the values on different pieces may differ.

Lemma 2.2. *Let $f : U \rightarrow \mathbb{C}$. Then*

$$f \text{ is locally constant on } U \iff \begin{array}{l} f \text{ is holomorphic on } U \\ \text{and } f'(z) = 0 \text{ for all } z \in U. \end{array} \quad (2.2)$$

Proof. The implication $(\implies)$ is clear: on a disc where f is constant, every difference quotient vanishes, so $f' \equiv 0$ there.

For $(\impliedby)$, go back to real analysis and apply the criterion

$$\begin{array}{l} F \text{ totally differentiable on } U \\ \text{and } D_{\mathbf{z}} = 0 \text{ for all } \mathbf{z} \in U \end{array} \implies F \text{ is locally constant around } \mathbf{z}. \quad (2.3)$$

Concretely: by Theorem 1.1, holomorphy of f gives total differentiability of F , and $f' = u_x + iv_x = 0$ together with the Cauchy–Riemann equations forces all four partial derivatives of u and v to vanish on U . Let $p, q \in D(z, \epsilon) \subseteq U$. Since a disc is convex, the curve

$$\phi(t) = p + t(q - p), \quad 0 \leq t \leq 1,$$

lies entirely in the disc and joins p to q . Define

$$g(t) = (u \circ \phi)(t) = u(p + t(q - p)).$$

Since $DF = 0$, its first row $\nabla u = (u_x, u_y)$ is zero. The multivariable chain rule and the identity $\phi'(t) = q - p$ therefore give

$$g'(t) = \nabla u(\phi(t)) \cdot \phi'(t) = \nabla u(\phi(t)) \cdot (q - p) = 0.$$

Thus g is constant by the mean value theorem, and hence $u(p) = g(0) = g(1) = u(q)$. The same argument applies to v . Therefore u and v are constant on the disc, and hence f is locally constant. $\square$

Example 2.3. Suppose $f : U \rightarrow \mathbb{C}$ is holomorphic and assumes *only real values*. Then f is locally constant.

Why? Write $f = u + iv$. Since f takes only real values, $v \equiv 0$. The Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x \quad (2.4)$$

then give $u_x = v_y = 0$ and $u_y = -v_x = 0$. Since

$$f' = u_x + iv_x, \quad (2.5)$$

we conclude $f' = 0$ on U , and therefore, by Lemma 2.2, f is locally constant.

3 Conformal maps

Let $U \subseteq \mathbb{C}$ be an open set, and let

$$\gamma : [a, b] \rightarrow U \quad (3.1)$$

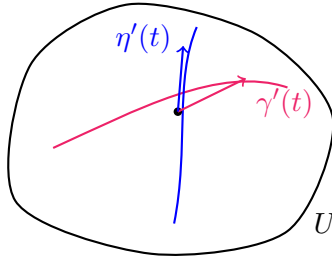
be a curve, written in coordinates as

$$\gamma(t) = x(t) + iy(t). \quad (3.2)$$

Assume γ is differentiable, i.e. $x'(t)$ and $y'(t)$ exist. Then

$$\gamma'(t) = x'(t) + iy'(t) \quad (3.3)$$

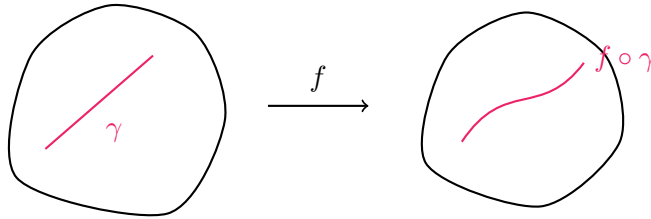
is the tangent vector to γ at the parameter t .



Remark. If $\gamma'(t) \neq 0$, then $(-y'(t), x'(t))$ is a normal vector to γ at t .

Now let $f : U \rightarrow \mathbb{C}$ be holomorphic on U . Composing, we obtain a new curve $f \circ \gamma$ in the image:

$$[a, b] \xrightarrow{\gamma} U \xrightarrow{f} \mathbb{C}, \quad t \mapsto f(\gamma(t)). \quad (3.4)$$



Example 3.1 (Chain rule). For a differentiable curve γ and f holomorphic,

$$\frac{d}{dt} f(\gamma(t)) = f'(\gamma(t)) \cdot \gamma'(t). \quad (3.5)$$

Proof: Regard f as the associated real map $F = (u, v) : U \rightarrow \mathbb{R}^2$ and apply the multivariable chain rule to $F \circ \gamma$. The Cauchy–Riemann equations identify the real derivative of F with multiplication by $f'(\gamma(t))$.

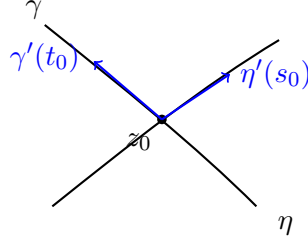
3.1 Angles between curves

Now suppose we are given two curves

$$\gamma : [a, b] \rightarrow U, \quad \eta : [c, d] \rightarrow U, \quad (3.6)$$

and assume that at a point of intersection

$$z_0 = \gamma(t_0) = \eta(s_0), \quad \gamma'(t_0) \neq 0, \quad \eta'(s_0) \neq 0. \quad (3.7)$$



Definition 3.2. The *oriented angle from γ to η at z_0* is the oriented angle from $\gamma'(t_0)$ to $\eta'(s_0)$, taken modulo 2π .

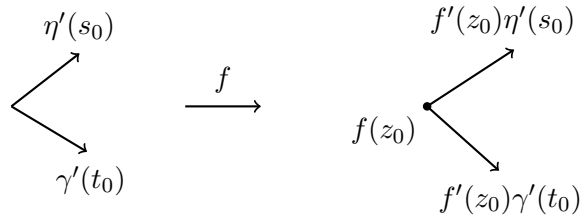
Lemma 3.3. Assume $f'(z_0) \neq 0$. Then the oriented angle from $f \circ \gamma$ to $f \circ \eta$ at $f(z_0)$ is equal to the oriented angle from γ to η at z_0 .

Remark. A map which preserves angles in this way is called *conformal*. So the lemma says: holomorphic $\implies$ conformal (at points where the derivative does not vanish).

Proof. By Example 3.1, the tangent vectors of the image curves at t_0 are

$$\left. \frac{d}{dt} f(\gamma(t)) \right|_{t_0} = f'(z_0) \gamma'(t_0), \quad \left. \frac{d}{ds} f(\eta(s)) \right|_{s_0} = f'(z_0) \eta'(s_0), \quad (3.8)$$

where $\gamma(t_0) = \eta(s_0) = z_0$.



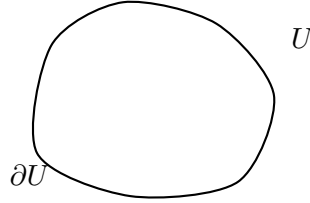
Now $f'(z_0)$ is a nonzero complex number. Writing $f'(z_0) = \rho e^{i\theta}$ with $\rho > 0$, multiplication by $f'(z_0)$ is a rotation by θ followed by a scaling by ρ . Both operations preserve the oriented angle from $\gamma'(t_0)$ to $\eta'(s_0)$. $\square$

4 Green's theorem and complex analysis

Theorem 4.1 (Green's theorem). *Let U be a bounded region in $\mathbb{R}^2$ whose positively oriented boundary ∂U is piecewise C^1 , and let $F = (P(x, y), Q(x, y))$ be a vector field whose components have continuous first partial derivatives on an open set containing $\bar{U}$. Then*

$$\int_{\partial U} P dx + Q dy = \iint_U \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy. \quad (4.1)$$

The left-hand side is the circulation integral of F along the boundary.



We first recall the definition of a complex line integral.

Definition 4.2. Let $\gamma : [a, b] \rightarrow U$ be a piecewise C^1 curve, $\gamma(t) = x(t) + iy(t)$. The *line integral of f along γ* is

$$\int_{\gamma} f(z) dz := \int_a^b f(\gamma(t)) \gamma'(t) dt. \quad (4.2)$$

Expanding into real and imaginary parts, with $f(\gamma(t))\gamma'(t) = (u + iv)(x' + iy')$, we obtain

$$\begin{aligned} \int_{\gamma} f(z) dz &= \int_a^b (u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t)) dt \\ &\quad + i \int_a^b (u(x(t), y(t))y'(t) + v(x(t), y(t))x'(t)) dt. \end{aligned}$$

Theorem 4.3 (Cauchy's theorem, first form). *Let $D \subseteq \mathbb{C}$ be a bounded region whose positively oriented boundary ∂D is piecewise C^1 . Suppose $f = u + iv$ is defined on an open set containing $\bar{D}$, is holomorphic there, and has continuous first partial derivatives. Then*

$$\boxed{\int_{\partial D} f(z) dz = 0.} \quad (4.3)$$

Proof. Suppose

$$G = (\underbrace{u(x, y)}_P, \underbrace{-v(x, y)}_Q). \quad (4.4)$$

Then by Green's theorem (Theorem 4.1),

$$\int_{\partial D} P dx + Q dy = \iint_D (-v_x - u_y) dx dy = 0 \quad (4.5)$$

by the Cauchy–Riemann equations, since $u_y = -v_x$.

On the other hand, by the expansion above,

$$\operatorname{Re} \left(\int_{\partial D} f(z) dz \right) = \int_{\partial D} P dx + Q dy = 0. \quad (4.6)$$

Similarly, taking $P = v$ and $Q = u$, Green’s theorem gives

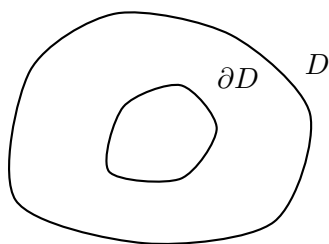
$$\operatorname{Im} \left(\int_{\partial D} f(z) dz \right) = \iint_D (u_x - v_y) dx dy = 0, \quad (4.7)$$

again by the Cauchy–Riemann equations, since $u_x = v_y$.

Therefore

$$\int_{\partial D} f(z) dz = 0. \quad (4.8)$$

□



Remark. This is a first version of *Cauchy’s theorem*: the integral of a holomorphic function around a closed curve bounding a region inside the domain vanishes. Note that the argument above needs the hypotheses of Green’s theorem, in particular continuity of the partial derivatives of u and v . Removing this extra hypothesis requires a different proof, which we will take up later.