

Contents

1	Review: complex differentiability and total differentiability	1
2	The Cauchy–Riemann criterion	3
3	Proof of the theorem	4
4	Examples	7

1 Review: complex differentiability and total differentiability

Definition 1.1. Let $f : U \rightarrow \mathbb{C}$, where $U \subseteq \mathbb{C}$ is open. The function f is *complex differentiable* at $w \in U$ if

$$\lim_{h \rightarrow 0} \frac{f(w+h) - f(w)}{h} \quad (1.1)$$

exists. This limit is denoted by $f'(w)$.

Equivalently, there exist a function σ defined in a small neighbourhood of $0 \in \mathbb{C}$ and a complex number $\xi \in \mathbb{C}$ such that

1. $f(w+h) - f(w) - \xi h = \sigma(h)h$;
2. $\lim_{h \rightarrow 0} \sigma(h) = 0$.

The neighbourhood is chosen small enough that $w+h \in U$. In this formulation $\xi = f'(w)$.

Definition 1.2. Now let F be a function on an open set $U \subseteq \mathbb{R}^2$, with

$$F : U \rightarrow \mathbb{R}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}. \quad (1.2)$$

The function F is said to be *totally differentiable* at the point $\mathbf{w} = \begin{pmatrix} x \\ y \end{pmatrix} \in U$ if there exists a real 2×2 matrix $D_{\mathbf{w}}$ such that

$$\lim_{\|\mathbf{h}\| \rightarrow 0} \frac{F(\mathbf{w} + \mathbf{h}) - F(\mathbf{w}) - D_{\mathbf{w}}\mathbf{h}}{\|\mathbf{h}\|} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (1.3)$$

Here $\mathbf{h}$ is a vector in $\mathbb{R}^2$, and $\|\mathbf{h}\|$ is its Euclidean norm.

Criterion from real analysis

The existence of all four first partial derivatives alone does not imply total differentiability, or even continuity.

Example 1.3. Define $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$F(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}, \quad u(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0), \end{cases} \quad v(x, y) = 0. \quad (1.4)$$

Away from the origin, all four partial derivatives exist because the defining functions are smooth. Since $u(t, 0) = u(0, t) = 0$ and $v = 0$, the difference quotients along the coordinate axes give

$$u_x(0, 0) = u_y(0, 0) = v_x(0, 0) = v_y(0, 0) = 0. \quad (1.5)$$

Thus all four partial derivatives exist everywhere. However, along the parabola $y = x^2$, for $t \neq 0$ we have

$$u(t, t^2) = \frac{t^4}{t^4 + t^4} = \frac{1}{2}. \quad (1.6)$$

Consequently $F(t, t^2) = (\frac{1}{2}, 0)$ does not tend to $F(0, 0) = (0, 0)$ as $t \rightarrow 0$. Thus F is not continuous at the origin, and hence cannot be totally differentiable there.

We can also check the definition directly. If F were totally differentiable at the origin, its derivative matrix would have to be the zero matrix: applying the definition along each coordinate axis forces both columns to vanish. But along $\mathbf{h} = (t, t^2)$, with $t \neq 0$, we have

$$\frac{\|F(t, t^2) - F(0, 0)\|}{\|(t, t^2)\|} = \frac{1}{2\sqrt{t^2 + t^4}} \rightarrow +\infty \quad (t \rightarrow 0). \quad (1.7)$$

This does not tend to zero. Hence F is not totally differentiable at $(0, 0)$.

The following criterion gives a sufficient condition.

Theorem 1.4 (Sufficient criterion for total differentiability). *If the first partial derivatives of u and v exist in a neighbourhood of $\mathbf{w}$ and are continuous at $\mathbf{w}$, then F is totally differentiable at $\mathbf{w}$, and its derivative matrix is the Jacobian*

$$D_{\mathbf{w}} = J_F(\mathbf{w}) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}_{\mathbf{w}}. \quad (1.8)$$

Note that continuity of the partial derivatives is a sufficient condition, but it is not a necessary condition for total differentiability at a point. Total differentiability at $\mathbf{w}$ does, however, imply that all first partial derivatives exist at $\mathbf{w}$ and form the matrix $D_{\mathbf{w}}$, even when those partial derivatives are not continuous there.

Example 1.5 (Total differentiability without continuous partial derivatives). Define

$$g(t) = \begin{cases} t^2 \sin(1/t), & t \neq 0, \\ 0, & t = 0, \end{cases} \quad F(x, y) = \begin{pmatrix} g(x) \\ g(y) \end{pmatrix}. \quad (1.9)$$

Since $|g(t)| \leq t^2$, for $\mathbf{h} = (x, y) \neq (0, 0)$ we have

$$\frac{\|F(x, y) - F(0, 0)\|}{\sqrt{x^2 + y^2}} \leq \frac{\sqrt{x^4 + y^4}}{\sqrt{x^2 + y^2}} \leq \sqrt{x^2 + y^2} \rightarrow 0. \quad (1.10)$$

Thus F is totally differentiable at the origin, with derivative matrix $D_{\mathbf{0}} = 0$.

However, $g'(0) = \lim_{t \rightarrow 0} t \sin(1/t) = 0$, while for $t \neq 0$,

$$g'(t) = 2t \sin(1/t) - \cos(1/t). \quad (1.11)$$

Writing $F = (u, v)$, all four partial derivatives exist everywhere:

$$u_x(x, y) = g'(x), \quad u_y(x, y) = 0, \quad v_x(x, y) = 0, \quad v_y(x, y) = g'(y). \quad (1.12)$$

For $t_n = 1/(2\pi n)$, $n \geq 1$, we have $t_n \rightarrow 0$ but $g'(t_n) = -1$. Hence $u_x(t_n, 0) = -1$ does not tend to $u_x(0, 0) = 0$, and $v_y(0, t_n) = -1$ does not tend to $v_y(0, 0) = 0$. Thus u_x and v_y are not continuous at the origin, even though F is totally differentiable there. This shows that the continuity hypothesis in Theorem 1.4 is sufficient but not necessary.

2 The Cauchy–Riemann criterion

What is the relation between total differentiability of F and complex differentiability of f ? Identify $\mathbb{C}$ with $\mathbb{R}^2$ via $x + iy \leftrightarrow (x, y)$, and let U be open. Write

$$f : U \rightarrow \mathbb{C}, \quad f(x + iy) = u(x, y) + iv(x, y), \quad F(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}. \quad (2.1)$$

The naive guess is

$$F \text{ is totally differentiable at } \mathbf{w} \iff f \text{ is complex differentiable at } w. \quad (2.2)$$

This guess is false. The correct statement is the following.

Theorem 2.1. *The function f is complex differentiable at $w = x + iy$ if and only if $F = \begin{pmatrix} u \\ v \end{pmatrix}$ is totally differentiable at $\mathbf{w} = \begin{pmatrix} x \\ y \end{pmatrix}$ and*

$$\boxed{u_x = v_y, \quad u_y = -v_x} \quad (2.3)$$

at (x, y) . These are the Cauchy–Riemann equations.

Remark. In light of Theorem 2.1 and Theorem 1.4, we obtain the following useful sufficient condition: f is complex differentiable at $w = a + ib$ if u_x, v_x, u_y, v_y all exist in a neighbourhood of (a, b) , are continuous at (a, b) , and satisfy the Cauchy–Riemann equations

$$u_x(a, b) = v_y(a, b), \quad u_y(a, b) = -v_x(a, b). \quad (2.4)$$

Equivalently, under the assumption that these partial derivatives exist in a neighbourhood of (a, b) and are continuous at (a, b) , f is complex differentiable at $a + ib$ if and only if the Cauchy–Riemann equations hold there. Continuity of the partial derivatives, together with the Cauchy–Riemann equations, is sufficient for complex differentiability at a point; continuity of the partial derivatives is not necessary.

3 Proof of the theorem

We prove Theorem 2.1. First suppose f is complex differentiable at $w = x + iy$, and write $f(x + iy) = u(x, y) + iv(x, y)$. We want to show that

$$F(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} \quad (3.1)$$

is totally differentiable at $\mathbf{w} = (x, y)$ and that the Cauchy–Riemann equations hold at (x, y) .

3.1 Why is the associated real function totally differentiable?

Since f is complex differentiable, there exist a function σ on a neighbourhood of 0 and a number $\alpha \in \mathbb{C}$ such that

$$f(w + h) - f(w) - \alpha h = \sigma(h)h, \quad \lim_{h \rightarrow 0} \sigma(h) = 0. \quad (3.2)$$

Set

$$\mathbf{w} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} r \\ s \end{pmatrix}, \quad h = r + is. \quad (3.3)$$

Then

$$F(\mathbf{w} + \mathbf{h}) - F(\mathbf{w}) = \begin{pmatrix} u(x + r, y + s) - u(x, y) \\ v(x + r, y + s) - v(x, y) \end{pmatrix}. \quad (3.4)$$

We seek a matrix $D_{\mathbf{w}}$ for which the components of $F(\mathbf{w} + \mathbf{h}) - F(\mathbf{w}) - D_{\mathbf{w}}\mathbf{h}$ are the real and imaginary parts of

$$f(w + h) - f(w) - \alpha h = \sigma(h)h. \quad (3.5)$$

Write $\alpha = a + ib$. Since $h = r + is$, we have

$$\begin{aligned} \alpha h &= (a + ib)(r + is) \\ &= ar + ias + ibr + i^2bs \\ &= (ar - bs) + i(as + br). \end{aligned}$$

The corresponding real matrix multiplication is

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} ar - bs \\ as + br \end{pmatrix}. \quad (3.6)$$

Therefore choose

$$D_{\mathbf{w}} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \quad (3.7)$$

By what we have just observed,

$$F(\mathbf{w} + \mathbf{h}) - F(\mathbf{w}) - D_{\mathbf{w}}\mathbf{h} = \begin{pmatrix} \operatorname{Re}(\sigma(h)h) \\ \operatorname{Im}(\sigma(h)h) \end{pmatrix}. \quad (3.8)$$

Further, $\|\mathbf{h}\| = \sqrt{r^2 + s^2} = |h|$, so

$$\begin{pmatrix} \operatorname{Re} \left(\frac{\sigma(h)h}{|h|} \right) \\ \operatorname{Im} \left(\frac{\sigma(h)h}{|h|} \right) \end{pmatrix} = \frac{F(\mathbf{w} + \mathbf{h}) - F(\mathbf{w}) - D_{\mathbf{w}}\mathbf{h}}{\|\mathbf{h}\|}. \quad (3.9)$$

By assumption, $\lim_{h \rightarrow 0} \sigma(h) = 0$, so we now need to consider

$$\lim_{h \rightarrow 0} \frac{\sigma(h)h}{|h|}. \quad (3.10)$$

For $h \neq 0$, write $h = |h|e^{i\theta_h}$. Thus

$$\frac{\sigma(h)h}{|h|} = \sigma(h)e^{i\theta_h}. \quad (3.11)$$

The number $e^{i\theta_h}$ lies on the unit circle. Consequently,

$$\left| \frac{\sigma(h)h}{|h|} \right| = |\sigma(h)| |e^{i\theta_h}| = |\sigma(h)| \rightarrow 0, \quad (3.12)$$

and hence

$$\lim_{h \rightarrow 0} \frac{\sigma(h)h}{|h|} = 0. \quad (3.13)$$

If this complex expression tends to zero, its real and imaginary parts also tend to zero:

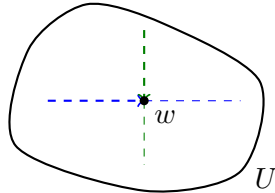
$$\operatorname{Re} \left(\frac{\sigma(h)h}{|h|} \right) \rightarrow 0, \quad \operatorname{Im} \left(\frac{\sigma(h)h}{|h|} \right) \rightarrow 0. \quad (3.14)$$

Both components tend to zero. This is exactly the definition of total differentiability of F at $\mathbf{w}$.

3.2 Why do the Cauchy–Riemann equations hold?

We now verify that the components u and v of the associated real function satisfy the Cauchy–Riemann equations at the point under consideration, $w = x + iy$. Write

$$f(w) = u(x, y) + iv(x, y). \quad (3.15)$$



First approach horizontally, so the complex increment is $h + i0$ with $h \in \mathbb{R}$. Then

$$\begin{aligned}
f'(w) &= \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} \frac{f(w+h) - f(w)}{h} \\
&= \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} \frac{u(x+h, y) + iv(x+h, y) - u(x, y) - iv(x, y)}{h} \\
&= \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} \frac{u(x+h, y) - u(x, y)}{h} + i \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} \frac{v(x+h, y) - v(x, y)}{h} \\
&= u_x(x, y) + iv_x(x, y).
\end{aligned}$$

Thus

$$\boxed{f'(w) = u_x + iv_x.} \quad (3.16)$$

Next approach along the vertical axis. The complex increment is now ih , where $h \in \mathbb{R}$. By the definition of the complex derivative, using this vertical approach gives the same limit:

$$f'(w) = \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} \frac{f(w+ih) - f(w)}{ih}. \quad (3.17)$$

Expanding this expression yields

$$\begin{aligned}
f'(w) &= \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} \frac{u(x, y+h) + iv(x, y+h) - u(x, y) - iv(x, y)}{ih} \\
&= \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} \frac{u(x, y+h) - u(x, y)}{ih} + \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} \frac{i(v(x, y+h) - v(x, y))}{ih} \\
&= \frac{1}{i} \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} \frac{u(x, y+h) - u(x, y)}{h} + \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} \frac{v(x, y+h) - v(x, y)}{h} \\
&= -iu_y(x, y) + v_y(x, y), \quad \text{since } \frac{1}{i} = -i.
\end{aligned}$$

Therefore

$$\boxed{f'(w) = -iu_y + v_y = u_x + iv_x.} \quad (3.18)$$

Equating real and imaginary parts gives

$$\boxed{u_x = v_y, \quad v_x = -u_y.} \quad (3.19)$$

This proves the forward implication: complex differentiability of f implies total differentiability of F and the Cauchy–Riemann equations. The reverse implication is proved next.

The reverse implication

Suppose F is totally differentiable at $\mathbf{w}$ and the Cauchy–Riemann equations hold there. All partial derivatives below are evaluated at (x, y) . The derivative matrix is

$$D_{\mathbf{w}} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad a = u_x = v_y, \quad b = v_x = -u_y. \quad (3.20)$$

Set $\alpha = a + ib$ and let

$$E(\mathbf{h}) = F(\mathbf{w} + \mathbf{h}) - F(\mathbf{w}) - D_{\mathbf{w}}\mathbf{h} = \begin{pmatrix} E_1(\mathbf{h}) \\ E_2(\mathbf{h}) \end{pmatrix}. \quad (3.21)$$

Total differentiability means $\|E(\mathbf{h})\|/\|\mathbf{h}\| \rightarrow 0$. For $h = r + is$ corresponding to $\mathbf{h} = \begin{pmatrix} r \\ s \end{pmatrix}$, the matrix calculation above gives

$$f(w + h) - f(w) - \alpha h = E_1(\mathbf{h}) + iE_2(\mathbf{h}). \quad (3.22)$$

For $h \neq 0$, divide by h and take absolute values:

$$\begin{aligned} \left| \frac{f(w + h) - f(w)}{h} - \alpha \right| &= \frac{|E_1(\mathbf{h}) + iE_2(\mathbf{h})|}{|h|} \\ &= \frac{\sqrt{E_1(\mathbf{h})^2 + E_2(\mathbf{h})^2}}{\|\mathbf{h}\|} \\ &= \frac{\|E(\mathbf{h})\|}{\|\mathbf{h}\|} \rightarrow 0. \end{aligned}$$

Thus $f'(w)$ exists and equals $\alpha = u_x + iv_x$.

4 Examples

4.1 The exponential function is entire

Let $z = x + iy$. We show that e^z is entire, that is, complex differentiable at every point of $\mathbb{C}$. By Theorems 1.4 and 2.1, it is sufficient to verify that the first partial derivatives exist and are continuous, and that u and v satisfy the Cauchy–Riemann equations.

$$\begin{aligned} e^z &= e^x e^{iy} \\ &= e^x (\cos y + i \sin y) \\ &= \underbrace{e^x \cos y}_{u(x,y)} + i \underbrace{e^x \sin y}_{v(x,y)}. \end{aligned}$$

The four partial derivatives are

$$\begin{aligned} u_x &= e^x \cos y, & v_x &= e^x \sin y, \\ u_y &= -e^x \sin y, & v_y &= e^x \cos y. \end{aligned}$$

They exist and are continuous everywhere, and

$$u_x = v_y, \quad u_y = -v_x. \quad (4.1)$$

The derivative matrix has precisely the indicated pattern:

$$J_F = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}. \quad (4.2)$$

Hence the theorem shows that e^z is entire. The derivative formula also gives

$$(e^z)' = u_x + iv_x = e^x \cos y + ie^x \sin y = e^z. \quad (4.3)$$

Warning: The Cauchy–Riemann equations alone do not suffice

Just because u and v satisfy the Cauchy–Riemann equations at a point does not mean that $f(z) = u(x, y) + iv(x, y)$ is complex differentiable there. For example, let

$$f(z) = \begin{cases} \frac{z^5}{|z|^4}, & z \neq 0, \\ 0, & z = 0. \end{cases} \quad (4.4)$$

This function is continuous at the origin, since $|f(z)| = |z|$. It satisfies the Cauchy–Riemann equations there but is not complex differentiable there.

Verification of the partial derivatives. For real $t \neq 0$, $|t|^4 = t^4$ and $i^5 = i$, so

$$f(t) = \frac{t^5}{|t|^4} = t, \quad f(it) = \frac{i^5 t^5}{|it|^4} = it. \quad (4.5)$$

Thus $u(t, 0) = t$, $v(t, 0) = 0$, $u(0, t) = 0$, $v(0, t) = t$, and $u(0, 0) = v(0, 0) = 0$. Consequently,

$$\begin{aligned} u_x(0, 0) &= \lim_{t \rightarrow 0} \frac{u(t, 0) - u(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{t}{t} = 1, \\ v_x(0, 0) &= \lim_{t \rightarrow 0} \frac{v(t, 0) - v(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{0}{t} = 0, \\ u_y(0, 0) &= \lim_{t \rightarrow 0} \frac{u(0, t) - u(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{0}{t} = 0, \\ v_y(0, 0) &= \lim_{t \rightarrow 0} \frac{v(0, t) - v(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{t}{t} = 1. \end{aligned}$$

Hence $u_x(0, 0) = v_y(0, 0)$ and $u_y(0, 0) = -v_x(0, 0)$.

For $h \neq 0$, the difference quotient is

$$\begin{aligned} \frac{f(0+h) - f(0)}{h} &= \frac{f(h)}{h} \\ &= \frac{h^5/|h|^4}{h} \\ &= \frac{h^4}{|h|^4}. \end{aligned}$$

Its limit as $h \rightarrow 0$ does not exist. To see this explicitly, write $h = re^{i\theta}$, where $r > 0$. Then

$$\frac{h^4}{|h|^4} = \frac{r^4 e^{4i\theta}}{r^4} = e^{4i\theta}. \quad (4.6)$$

Along the positive real axis ($\theta = 0$), this quotient equals 1. Along the ray $\theta = \pi/4$, it equals $e^{i\pi} = -1$. Both paths approach the origin as $r \rightarrow 0$, but give different limits. Therefore $f'(0)$ does not exist.