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1 Complex Differentiability

Let $U \subseteq \mathbb{C}$ be open and let $f : U \rightarrow \mathbb{C}$.

Definition 1.1. We say that f is *complex differentiable* at $z_0 \in U$ if the limit

$$\lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{C} \setminus \{0\}}} \frac{f(z_0 + h) - f(z_0)}{h} \quad (1.1)$$

exists. (Since U is open, $z_0 + h \in U$ for all sufficiently small h , so the quotient makes sense.) When the limit exists it is denoted by $f'(z_0)$ and is called the *derivative* of f at z_0 .

If f is complex differentiable at every point of the open set U , we say that f is *holomorphic* on U .

The condition $h \rightarrow 0$ is a limit in the complex plane: h may approach 0 from any direction. This makes complex differentiability much more restrictive than ordinary one-variable differentiability, as the second example below shows.

Example 1.2 (The identity map). For $f(z) = z$ and every $z_0 \in \mathbb{C}$,

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{z_0 + h - z_0}{h} = 1.$$

Thus f is holomorphic on $\mathbb{C}$ and $f'(z) = 1$.

Example 1.3 (Complex conjugation). Let $f(z) = \bar{z}$. At any $z_0 \in \mathbb{C}$,

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{\bar{h}}{h}.$$

Writing $h = re^{i\theta}$ gives

$$\frac{\bar{h}}{h} = e^{-2i\theta}.$$

The value depends only on the direction of approach, and it varies with that direction: along the real axis ($\theta = 0$) the quotient is 1, while along the imaginary axis ($\theta = \pi/2$) it is -1 . Hence the limit does not exist, and complex conjugation is nowhere complex differentiable.

Note that as a map of $\mathbb{R}^2$, $(x, y) \mapsto (x, -y)$ is linear, hence as smooth as a function can be. Complex differentiability is a different, stronger, condition. In Lecture 2 we computed for this function that $u_x = 1 \neq -1 = v_y$; we will see in Section 4 that this is exactly the obstruction.

Exercise 1.4. Let $f(z) = |z|^2 = z\bar{z}$. Show that f is complex differentiable at $z_0 = 0$, with $f'(0) = 0$, but at no other point of $\mathbb{C}$. In particular f is not holomorphic on any open set, which is why the definition of “holomorphic” insists on differentiability throughout an open set.

2 Basic Properties

Proposition 2.1. *If f is complex differentiable at z_0 , then f is continuous at z_0 .*

Proof. For $z \neq z_0$ we have

$$f(z) - f(z_0) = \frac{f(z) - f(z_0)}{z - z_0} (z - z_0). \quad (2.1)$$

As $z \rightarrow z_0$, the first factor tends to $f'(z_0)$ and the second tends to 0. Therefore $f(z) \rightarrow f(z_0)$. $\square$

Proposition 2.2 (Differentiation rules). *Suppose f and g are complex differentiable at z_0 and let $c \in \mathbb{C}$. Then cf , $f + g$ and fg are complex differentiable at z_0 , and*

$$\begin{aligned} (cf)'(z_0) &= c f'(z_0), \\ (f + g)'(z_0) &= f'(z_0) + g'(z_0), \\ (fg)'(z_0) &= f'(z_0)g(z_0) + f(z_0)g'(z_0). \end{aligned}$$

If moreover $g(z_0) \neq 0$, then $g \neq 0$ on a neighbourhood of z_0 (by continuity), the quotient f/g is complex differentiable at z_0 , and

$$\left(\frac{f}{g}\right)'(z_0) = \frac{g(z_0)f'(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}. \quad (2.2)$$

Proof. The proofs are word for word the same as for real functions of one variable; they use only the algebra of limits and the continuity of f and g at z_0 . For example, try the proof of the product rule for yourself. (Hint: add and subtract $f(z_0 + h)g(z_0)$ in the numerator.) $\square$

Corollary 2.3. *Every polynomial $p(z) = a_n z^n + \cdots + a_1 z + a_0$ is holomorphic on $\mathbb{C}$, with*

$$p'(z) = n a_n z^{n-1} + \cdots + 2 a_2 z + a_1. \quad (2.3)$$

Every rational function p/q is holomorphic on the open set $\{z \in \mathbb{C} : q(z) \neq 0\}$.

Proof. Constants and the identity map are holomorphic on $\mathbb{C}$ (with derivatives 0 and 1), so by induction on n and the product rule, $(z^n)' = n z^{n-1}$. The rest follows from Proposition 2.2. $\square$

Theorem 2.4 (Chain rule). *Let $U, V \subseteq \mathbb{C}$ be open. Suppose $f : U \rightarrow V$ is complex differentiable at $z \in U$ and $g : V \rightarrow \mathbb{C}$ is complex differentiable at $f(z)$. Then $g \circ f$ is complex differentiable at z , and*

$$(g \circ f)'(z) = g'(f(z)) f'(z). \quad (2.4)$$

To prove the chain rule cleanly, it is useful to reformulate the definition of differentiability.

Lemma 2.5 (Linear approximation). *A function f is complex differentiable at z if and only if there are $\alpha \in \mathbb{C}$ and a function φ , defined on a neighbourhood of 0, such that*

$$f(z+h) - f(z) = \alpha h + h \varphi(h) \quad \text{for all small } h, \text{ including } h = 0, \quad (2.5)$$

and φ is continuous at 0 with $\varphi(0) = 0$. In this case $\alpha = f'(z)$.

Proof. If f is differentiable at z , put $\alpha = f'(z)$, and for $h \neq 0$ define

$$\varphi(h) = \frac{f(z+h) - f(z)}{h} - f'(z), \quad (2.6)$$

and set $\varphi(0) = 0$. Then (2.5) holds for all small h (trivially at $h = 0$), and $\varphi(h) \rightarrow 0 = \varphi(0)$ by the definition of $f'(z)$. Conversely, dividing (2.5) by $h \neq 0$ and letting $h \rightarrow 0$ shows that the difference quotient tends to α . $\square$

Proof of the chain rule. Put $w = f(z)$ and $k = f(z+h) - f(z)$, so that $f(z+h) = w + k$. Applying the lemma to f at z and to g at w , we get functions φ, ψ , each continuous at 0 and vanishing there, with

$$\begin{aligned} k &= f'(z) h + h \varphi(h), \\ g(w+k) - g(w) &= g'(w) k + k \psi(k). \end{aligned}$$

The second identity holds for all small k , *including* $k = 0$; this matters, because k can vanish for $h \neq 0$ (for instance if f is constant). Since f is continuous at z , $k \rightarrow 0$ as $h \rightarrow 0$, and hence $\psi(k) \rightarrow \psi(0) = 0$. Also $k/h \rightarrow f'(z)$. Therefore, for $h \neq 0$,

$$\begin{aligned} \frac{g(f(z+h)) - g(f(z))}{h} &= g'(w) \frac{k}{h} + \frac{k}{h} \psi(k) \\ &\rightarrow g'(w) f'(z) + f'(z) \cdot 0 = g'(f(z)) f'(z). \end{aligned}$$

This is the desired formula. $\square$

Remark. The familiar “proof” from calculus, which writes

$$\frac{g(f(z+h)) - g(f(z))}{h} = \frac{g(w+k) - g(w)}{k} \cdot \frac{k}{h},$$

is not valid as written: the first factor is undefined whenever $k = 0$. The linear approximation lemma is precisely the device that avoids dividing by k .

3 Real Total Differentiability

Write $z = x + iy$ and

$$f(z) = u(x, y) + i v(x, y). \quad (3.1)$$

Identifying $\mathbb{C}$ with $\mathbb{R}^2$, the function f corresponds to

$$F : U \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad F(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}. \quad (3.2)$$

Definition 3.1. The map F is *totally differentiable* at $\mathbf{a} \in U$ if there is a real 2×2 matrix $D_{\mathbf{a}}$ such that

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{\|F(\mathbf{a} + \mathbf{h}) - F(\mathbf{a}) - D_{\mathbf{a}}\mathbf{h}\|}{\|\mathbf{h}\|} = 0. \quad (3.3)$$

The matrix $D_{\mathbf{a}}$ is unique and is called the derivative, or Jacobian matrix, of F at $\mathbf{a}$.

If F is totally differentiable at $\mathbf{a}$, then taking $\mathbf{h} = t\mathbf{e}_1$ and $\mathbf{h} = t\mathbf{e}_2$ with $t \rightarrow 0$ shows that the first partial derivatives of u and v exist at $\mathbf{a}$ and that the derivative matrix is necessarily

$$D_{\mathbf{a}} = J_F(\mathbf{a}) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}_{\mathbf{a}}. \quad (3.4)$$

More generally, the directional derivative of F at $\mathbf{a}$ in the direction $\mathbf{h}$ is then $D_{\mathbf{a}}\mathbf{h}$. The converse fails: the existence of all directional derivatives at $\mathbf{a}$ does not by itself imply total differentiability at $\mathbf{a}$.

Exercise 3.2. Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by $F(0, 0) = (0, 0)$ and

$$F(x, y) = \left(\frac{xy^2}{x^2 + y^4}, 0 \right) \quad \text{for } (x, y) \neq (0, 0).$$

Show that every directional derivative of F at the origin exists (and that $J_F(0, 0)$ is the zero matrix), but that F is not even continuous at the origin. (Consider the curve $x = y^2$.)

Remark. A standard criterion from real analysis says that if the first partial derivatives of u and v exist in a neighbourhood of $\mathbf{a}$ and are continuous at $\mathbf{a}$, then F is totally differentiable at $\mathbf{a}$.

4 The Cauchy–Riemann Criterion

Complex differentiability is real total differentiability with an extra compatibility condition.

Theorem 4.1. Let $f = u + iv$ and let $F = (u, v)$. Then f is complex differentiable at $z = x + iy$ if and only if F is totally differentiable at (x, y) and

$$\boxed{u_x = v_y, \quad u_y = -v_x} \quad (4.1)$$

hold at (x, y) . These are the Cauchy–Riemann equations. In that case

$$f'(z) = u_x + i v_x = v_y - i u_y. \quad (4.2)$$

Remark (What the Cauchy–Riemann equations mean). The Cauchy–Riemann equations say exactly that the Jacobian matrix has the special form

$$J_F = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad a = u_x, \quad b = v_x. \quad (4.3)$$

Under the identification $\mathbb{C} \cong \mathbb{R}^2$, multiplication by the complex number $a + ib$ is the real-linear map with precisely this matrix. So Theorem 4.1 can be summarized as follows: *f is complex differentiable at z if and only if F is totally differentiable at z and its derivative D_z is not merely $\mathbb{R}$ -linear but $\mathbb{C}$ -linear, namely multiplication by $f'(z)$.*

Exercise 4.2. Using the partial derivatives computed in Lecture 2, check that $f(z) = z^2$ satisfies the Cauchy–Riemann equations at every point and that $f(z) = \bar{z}$ satisfies them nowhere. Then compute u, v for $f(z) = |z|^2$ and show that the equations hold only at $z = 0$. Compare with the first section of this lecture, and, for z^2 , verify that $u_x + i v_x$ agrees with the derivative $2z$ from Corollary 2.3.