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1 Review and Stereographic Projection

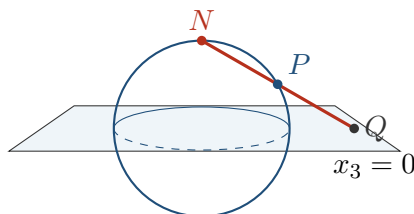
Last time we ended with a discussion of the extended complex plane

$$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}. \quad (1.1)$$

together with the stereographic projection

$$\pi : S^2 \setminus \{N\} \longrightarrow \mathbb{C}, \quad P \longmapsto Q, \quad (1.2)$$

defined by the following figure:



In coordinates, identifying the x_1x_2 -plane with $\mathbb{C}$, if $P = (x_1, x_2, x_3) \in S^2$, the point Q corresponds to

$$\boxed{z = \frac{x_1 + ix_2}{1 - x_3}}, \quad P = (x_1, x_2, x_3). \quad (1.3)$$

Geometrically, it is clear that π is a bijection. It extends to a bijection

$$\widehat{\pi} : S^2 \longrightarrow \widehat{\mathbb{C}} \quad \text{by setting} \quad \widehat{\pi}(N) = \infty. \quad (1.4)$$

Exercise 1.1. Given $z = a + ib \in \mathbb{C}$, prove that the preimage of z under π is the point

$$\pi^{-1}(z) = \left(\frac{2a}{1 + |z|^2}, \frac{2b}{1 + |z|^2}, \frac{|z|^2 - 1}{1 + |z|^2} \right) \in S^2. \quad (1.5)$$

Using stereographic projection, we can transport natural operations on S^2 to $\hat{\mathbb{C}}$, or vice versa. For example:

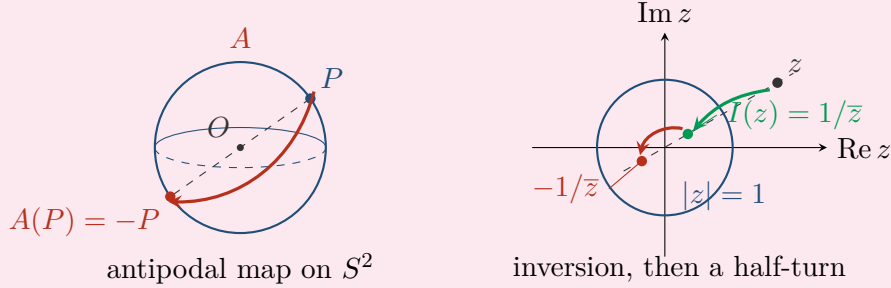
Exercise 1.2. Show that under stereographic projection the antipodal map

$$A : S^2 \longrightarrow S^2, \quad (x_1, x_2, x_3) \longmapsto (-x_1, -x_2, -x_3), \quad (1.6)$$

corresponds to the map

$$z \longmapsto -\frac{1}{\bar{z}} \quad (1.7)$$

on $\hat{\mathbb{C}}$, which is inversion in the unit circle followed by a half-turn.



By a circle in S^2 we mean the intersection of a plane in $\mathbb{R}^3$ with S^2 .

Exercise 1.3. Show that under stereographic projection, circles on S^2 are sent to either circles or lines in $\hat{\mathbb{C}}$. When does a circle in S^2 flatten out to a line? Can you get every circle or line in $\mathbb{C}$ from stereographic projection?

2 Functions of a Complex Variable

Let $S \subseteq \mathbb{C}$. A function of a complex variable is a map

$$f : S \longrightarrow \mathbb{C}, \quad z \longmapsto w = f(z). \quad (2.1)$$

Write

$$z = x + iy \quad \text{and} \quad w = u + iv. \quad (2.2)$$

Then a complex-valued function can be expressed as

$$f(z) = u(x, y) + i v(x, y), \quad (2.3)$$

where u and v are real-valued functions of the two real variables x and y .

Example 2.1 (The squaring map). For $f(z) = z^2$,

$$f(x + iy) = (x + iy)^2 = (x^2 - y^2) + i(2xy). \quad (2.4)$$

Hence

$$u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy. \quad (2.5)$$

Their first partial derivatives are

$$u_x = 2x, \quad u_y = -2y, \quad v_x = 2y, \quad v_y = 2x. \quad (2.6)$$

Example 2.2 (Complex conjugation). For $f(z) = \bar{z}$,

$$f(x + iy) = x - iy, \quad u(x, y) = x, \quad v(x, y) = -y. \quad (2.7)$$

Thus

$$u_x = 1, \quad u_y = 0, \quad v_x = 0, \quad v_y = -1. \quad (2.8)$$

Example 2.3 (Power maps). For $f(z) = z^n$, where n is a positive integer, write $z = re^{i\theta}$. Then

$$f(z) = r^n e^{in\theta} = r^n \cos(n\theta) + i r^n \sin(n\theta). \quad (2.9)$$

Consequently,

$$u(r, \theta) = r^n \cos(n\theta), \quad v(r, \theta) = r^n \sin(n\theta). \quad (2.10)$$

The corresponding Cartesian formulas can be obtained by substituting $r = \sqrt{x^2 + y^2}$ and an argument θ of $x + iy$.

The relations

$$u_x = v_y, \quad u_y = -v_x, \quad (2.11)$$

are the *Cauchy–Riemann equations*. The squaring map satisfies them, whereas complex conjugation does not. We will study their significance later.

Exercise 2.4. Let

$$z = re^{i\theta} \quad \text{and} \quad f(z) = u(r, \theta) + i v(r, \theta), \quad (2.12)$$

where $r > 0$. Show that the Cartesian Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x \quad (2.13)$$

are equivalent to the polar Cauchy–Riemann equations

$$\boxed{u_r = \frac{1}{r} v_\theta, \quad v_r = -\frac{1}{r} u_\theta}. \quad (2.14)$$

Then verify directly that $f(z) = z^n$, with

$$u(r, \theta) = r^n \cos(n\theta), \quad v(r, \theta) = r^n \sin(n\theta), \quad (2.15)$$

satisfies these equations.

3 Visualizing Complex-Valued Functions

The graph of a function $f : \mathbb{C} \rightarrow \mathbb{C}$ naturally lives in four real dimensions, so it cannot be drawn in an ordinary three-dimensional plot. Instead, we study how familiar subsets of the z -plane are transformed into subsets of the w -plane.

Images of Vertical Lines: $w = z^2$

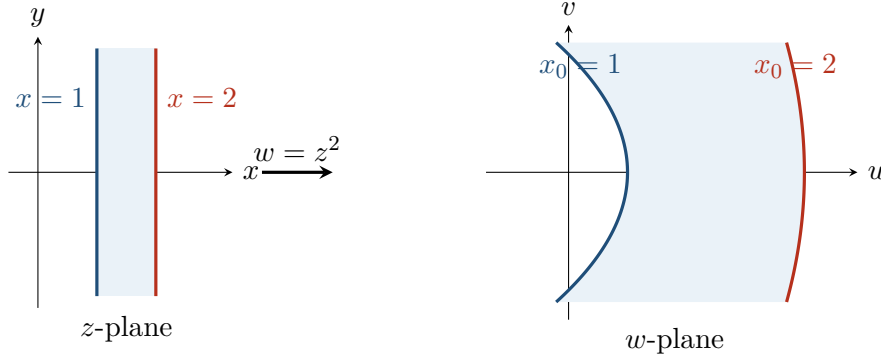
For the squaring map $w = z^2$, consider the vertical line $x = x_0$. Since

$$u = x_0^2 - y^2, \quad v = 2x_0y, \quad (3.1)$$

we may eliminate y (when $x_0 \neq 0$) to obtain

$$u = x_0^2 - \frac{v^2}{4x_0^2}. \quad (3.2)$$

Thus vertical lines in the z -plane are mapped to parabolas in the w -plane.



In particular, for $x_0 = 1$ the image parabola is

$$u = 1 - \left(\frac{v}{2}\right)^2, \quad (3.3)$$

while for $x_0 = 2$ it is

$$u = 4 - \left(\frac{v}{4}\right)^2. \quad (3.4)$$

The strip $1 < x < 2$ maps to the region between these two parabolas.

Images of Horizontal Lines: $w = z^2$

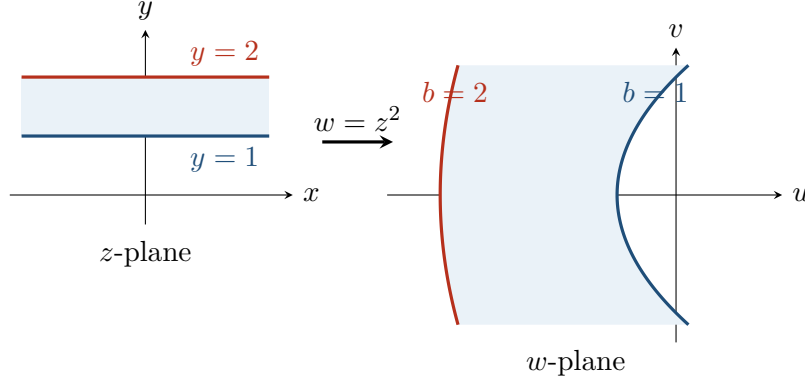
Now consider the horizontal line $y = b$. Writing $z = x + ib$ and $w = u + iv = z^2$, we have

$$u = x^2 - b^2, \quad v = 2bx. \quad (3.5)$$

If $b \neq 0$, eliminating x gives

$$\boxed{u = \frac{v^2}{4b^2} - b^2}. \quad (3.6)$$

Thus a horizontal line maps to a parabola with vertex $(-b^2, 0)$, axis the real axis, and opening to the right. The lines $y = b$ and $y = -b$ have the same image, traced with opposite orientations.

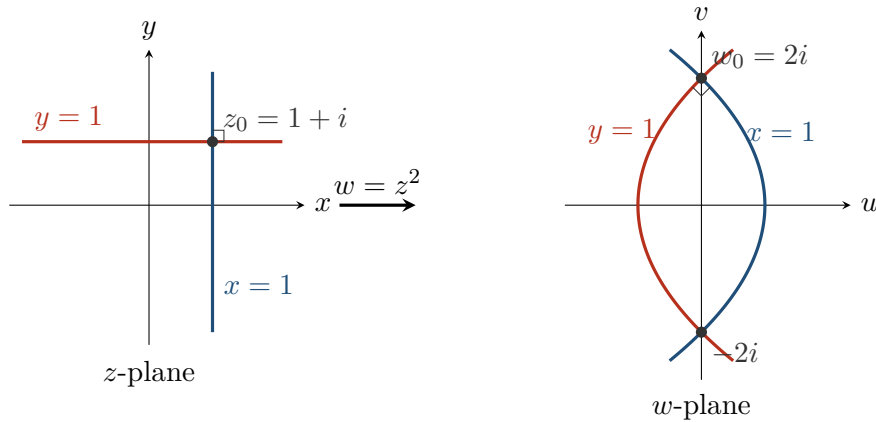


For example, $y = 1$ maps to $u = v^2/4 - 1$, while $y = 2$ maps to $u = v^2/16 - 4$. The strip $1 < y < 2$ maps to the region between these two parabolas. When $b = 0$, the real axis maps to the nonnegative real axis, since $w = x^2 \geq 0$.

Intersecting Horizontal and Vertical Lines

Consider the lines $x = 1$ and $y = 1$, which intersect at $z_0 = 1 + i$. Their images under $w = z^2$ are respectively

$$u = 1 - \frac{v^2}{4} \quad \text{and} \quad u = \frac{v^2}{4} - 1. \quad (3.7)$$



The upper intersection is $w_0 = z_0^2 = 2i$. The right angle is preserved because $z \mapsto z^2$ is conformal at $z_0 \neq 0$. The second intersection, at $-2i$, occurs because the points $1 - i$ on $x = 1$ and $-1 + i$ on $y = 1$ are negatives of one another and therefore have the same square.

4 Basic Topology in the Complex Plane

Definition 4.1. For $\alpha \in \mathbb{C}$ and $r > 0$, the *open disc* centered at α with radius r is

$$D(\alpha, r) = \{z \in \mathbb{C} : |z - \alpha| < r\}. \quad (4.1)$$

The corresponding *closed disc* is

$$\overline{D}(\alpha, r) = \{z \in \mathbb{C} : |z - \alpha| \leq r\}. \quad (4.2)$$

The open unit disc is denoted by

$$\mathbb{D} = D(0, 1). \quad (4.3)$$

Definition 4.2. A subset $U \subseteq \mathbb{C}$ is *open* if, for every $\alpha \in U$, there is an $r > 0$ such that

$$D(\alpha, r) \subseteq U. \quad (4.4)$$

A subset $F \subseteq \mathbb{C}$ is *closed* if its complement $\mathbb{C} \setminus F$ is open.

Definition 4.3. Let $S \subseteq \mathbb{C}$ and $\alpha \in \mathbb{C}$.

- The point α is an *interior point* of S if $D(\alpha, r) \subseteq S$ for some $r > 0$.
- The point α is a *boundary point* of S if, for every $r > 0$, the disc $D(\alpha, r)$ meets both S and $\mathbb{C} \setminus S$.
- The point α is an *adherent point* of S if every disc $D(\alpha, r)$ meets S .

Every interior point and every boundary point of S is adherent to S . Indeed, the set of adherent points is the closure of S , which is the union of the interior of S and its boundary.

Definition 4.4. Let $S \subseteq \mathbb{C}$, let $f : S \rightarrow \mathbb{C}$, and suppose α is adherent to $S \setminus \{\alpha\}$. We say that

$$\lim_{\substack{z \rightarrow \alpha \\ z \in S}} f(z) = w \quad (4.5)$$

if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $z \in S$,

$$0 < |z - \alpha| < \delta \implies |f(z) - w| < \varepsilon. \quad (4.6)$$

This is the same ε - δ definition used for functions of a real variable, with distance measured by the complex modulus.