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1 Basics of Complex Numbers

Definition 1.1. The set of *complex numbers* is

$$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}, \quad (1.1)$$

where $i^2 = -1$. For $z = a + ib$, we write

$$\operatorname{Re}(z) = a \quad \text{and} \quad \operatorname{Im}(z) = b. \quad (1.2)$$

The set $\mathbb{C}$ comes with the following algebraic operations. If $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$, and $c \in \mathbb{R}$, then

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2), \quad (1.3)$$

$$cz_1 = ca_1 + i(cb_1), \quad (1.4)$$

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) \quad (1.5)$$

$$= (a_1 a_2 - b_1 b_2) + i(b_1 a_2 + a_1 b_2). \quad (1.6)$$

The last identity follows by expanding and using $i^2 = -1$.

The additive identity is 0, and the multiplicative identity is 1:

$$0 + z = z + 0 = z, \quad 1 \cdot z = z \cdot 1 = z. \quad (1.7)$$

We regard $\mathbb{R}$ as a subset of $\mathbb{C}$ by identifying $a \in \mathbb{R}$ with $a + i0 \in \mathbb{C}$.

Definition 1.2. If $z = a + ib$, its *complex conjugate* is

$$\bar{z} = a - ib. \quad (1.8)$$

Multiplying a complex number by its conjugate gives

$$z\bar{z} = (a + ib)(a - ib) = a^2 + b^2 \in \mathbb{R}_{\geq 0}. \quad (1.9)$$

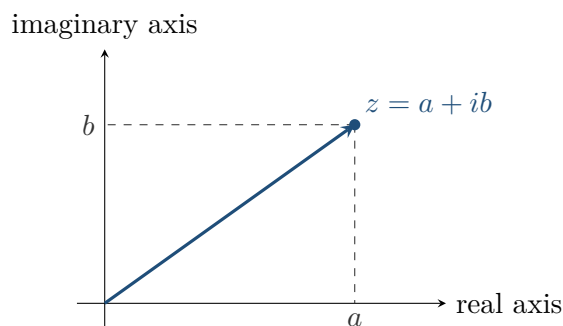
If $z \neq 0$, then $z\bar{z} = a^2 + b^2 > 0$, and hence

$$z^{-1} = \frac{\bar{z}}{a^2 + b^2}. \quad (1.10)$$

Thus every nonzero complex number has a multiplicative inverse. In particular, $\mathbb{C}$ is a field, much like $\mathbb{R}$.

The Complex Plane and the Modulus

We identify $\mathbb{C}$ with $\mathbb{R}^2$ by sending $z = a + ib$ to the point (a, b) . Under this identification, the real part is the horizontal coordinate and the imaginary part is the vertical coordinate.



Definition 1.3. The *modulus* (or absolute value) of $z = a + ib$ is

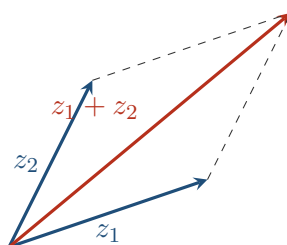
$$|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}. \quad (1.11)$$

Geometrically, $|z|$ is the length of the vector in $\mathbb{R}^2$ represented by z .

The triangle inequality in $\mathbb{R}^2$ becomes the triangle inequality in $\mathbb{C}$:

$$|z_1 + z_2| \leq |z_1| + |z_2|. \quad (1.12)$$

Addition of complex numbers is simply vector addition.



Polar Form

Let $z = a + ib \neq 0$. Write $r = |z|$ and let θ be the angle measured counterclockwise from the positive real axis to the vector representing z . Then

$$a = r \cos \theta, \quad b = r \sin \theta. \quad (1.13)$$

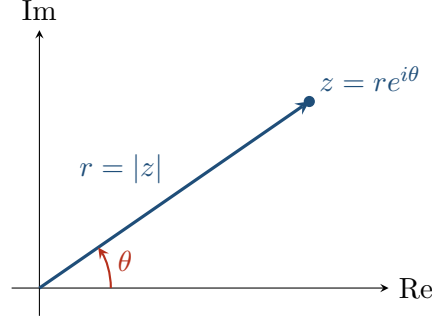
Euler's formula is

$$e^{i\theta} = \cos \theta + i \sin \theta. \quad (1.14)$$

It allows us to write z in *polar form*:

$$\boxed{z = re^{i\theta}} = r \cos \theta + ir \sin \theta. \quad (1.15)$$

Here $r = |z|$ is the modulus of z , and θ is an argument of z .



Every $z \in \mathbb{C} \setminus \{0\}$ can be written as $z = re^{i\theta}$ with $r > 0$ and $\theta \in \mathbb{R}$. The modulus r is unique, while the argument is unique only modulo 2π :

$$e^{i(\theta+2\pi k)} = e^{i\theta} \quad (k \in \mathbb{Z}). \quad (1.16)$$

If we require $\theta \in [0, 2\pi)$, then the argument is uniquely specified.

Multiplication in Polar Form

Suppose

$$z_1 = re^{i\theta} \quad \text{and} \quad z_2 = se^{i\varphi}. \quad (1.17)$$

Then

$$z_1 z_2 = (rs)e^{i(\theta+\varphi)}. \quad (1.18)$$

Thus multiplication of complex numbers *multiplies moduli and adds arguments*.

To verify this directly, expand using Euler's formula:

$$e^{i\theta} e^{i\varphi} = (\cos \theta + i \sin \theta)(\cos \varphi + i \sin \varphi) \quad (1.19)$$

$$= (\cos \theta \cos \varphi - \sin \theta \sin \varphi) \quad (1.20)$$

$$+ i(\sin \theta \cos \varphi + \cos \theta \sin \varphi) \quad (1.21)$$

$$= \cos(\theta + \varphi) + i \sin(\theta + \varphi) \quad (1.22)$$

$$= e^{i(\theta+\varphi)}. \quad (1.23)$$

The Complex Exponential

Definition 1.4. For $z = a + ib \in \mathbb{C}$, define

$$e^z = e^a e^{ib}. \quad (1.24)$$

The familiar exponential law continues to hold: for all $\alpha, \beta \in \mathbb{C}$,

$$e^\alpha e^\beta = e^{\alpha+\beta}. \quad (1.25)$$

Verify this identity as an exercise.

Remark. For complex z , the number e^z need not be real or positive. In fact, e^{ib} lies on the unit circle.

The n -th Roots of a Complex Number

Example 1.5. Let $z \in \mathbb{C} \setminus \{0\}$ and let n be a positive integer. We solve

$$w^n = z. \quad (1.26)$$

Write

$$w = te^{i\varphi} \quad \text{and} \quad z = re^{i\theta}, \quad (1.27)$$

where $t, r > 0$. Then

$$w^n = t^n e^{in\varphi} = re^{i\theta}. \quad (1.28)$$

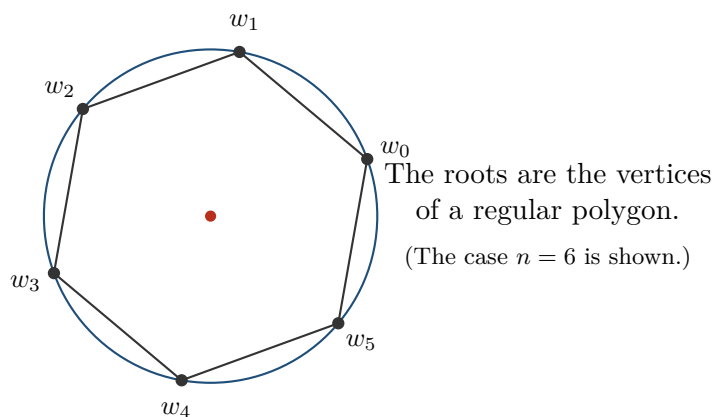
Comparing moduli gives $t^n = r$, so $t = r^{1/n}$. Comparing arguments modulo 2π gives

$$n\varphi = \theta + 2\pi k, \quad \text{so} \quad \varphi = \frac{\theta}{n} + \frac{2\pi k}{n}. \quad (1.29)$$

Consequently, the n distinct solutions are

$$\boxed{w_k = r^{1/n} e^{i\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)}}, \quad k = 0, 1, \dots, n-1. \quad (1.30)$$

The points $w_0, \dots, w_{n-1}$ are equally spaced on the circle of radius $r^{1/n}$, so they form the vertices of a regular n -gon.



2 Stereographic Projection

It is often useful to add one point, denoted by ∞ , to the complex plane.

Definition 2.1. The *extended complex plane* is

$$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}. \quad (2.1)$$

For $a, b \in \mathbb{C}$, we use the conventions

$$a + \infty = \infty + a = \infty, \quad b \cdot \infty = \infty \cdot b = \infty \quad (b \neq 0), \quad (2.2)$$

and

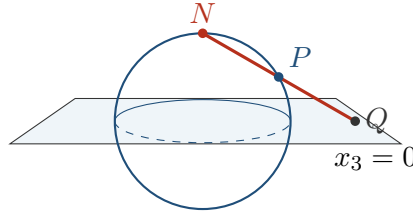
$$\frac{a}{0} = \infty \quad (a \neq 0), \quad \frac{b}{\infty} = 0. \quad (2.3)$$

Expressions such as $\infty - \infty$, $0 \cdot \infty$, ∞/∞ , and $0/0$ remain undefined.

Now let

$$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\} \quad (2.4)$$

be the unit sphere, and let $N = (0, 0, 1)$ be its north pole. For $P \in S^2 \setminus \{N\}$, draw the line through N and P . It intersects the plane $x_3 = 0$ in a unique point Q .



Identifying the x_1x_2 -plane with $\mathbb{C}$, the point Q corresponds to

$$\boxed{z = \frac{x_1 + ix_2}{1 - x_3}}, \quad P = (x_1, x_2, x_3). \quad (2.5)$$

To verify this formula, parametrize the line through N and P by

$$\ell_P(t) = N + t(P - N) = (tx_1, tx_2, 1 + t(x_3 - 1)). \quad (2.6)$$

This line meets the plane $x_3 = 0$ when $1 + t(x_3 - 1) = 0$, so $t = 1/(1 - x_3)$. Therefore

$$Q = \ell_P\left(\frac{1}{1 - x_3}\right) = \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3}, 0\right), \quad (2.7)$$

which corresponds to the displayed complex number z .

Definition 2.2. The map

$$\pi : S^2 \setminus \{N\} \longrightarrow \mathbb{C}, \quad P \longmapsto z, \quad (2.8)$$

is called *stereographic projection*.

The map π is a bijection. It extends to a bijection

$$\hat{\pi} : S^2 \longrightarrow \hat{\mathbb{C}} \quad \text{by setting} \quad \hat{\pi}(N) = \infty. \quad (2.9)$$

Thus the extended complex plane may be viewed as a sphere. Under stereographic projection, circles on S^2 become circles or lines in $\hat{\mathbb{C}}$; circles through N become lines in $\mathbb{C}$.