

PROBLEM SET 3
MATH 411: HONORS COMPLEX VARIABLES, FALL 2026

Suggested completion date: Monday, October 26, 2026.

These are practice problems and are not to be turned in. Justify your answers and all limiting arguments. Throughout, homotopies of paths *fix their endpoints*, and a simply connected open set is required to be nonempty and connected.

- (1) **Real integrals by contour integration.** Use Cauchy's theorem to evaluate the following improper integrals.

(a)

$$\int_0^\infty \cos(t^2) dt.$$

Hint: Integrate e^{-z^2} around the boundary of the sector $0 \leq \arg z \leq \pi/4$, $|z| \leq R$. You may use

$$\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}, \quad \sin u \geq \frac{2u}{\pi} \quad (0 \leq u \leq \pi/2).$$

Show that the integral over the circular arc tends to zero; a bound using only the maximum of the integrand on the arc will not suffice.

(b) For $a, b > 0$, evaluate

$$\int_0^\infty e^{-ax} \cos(bx) dx.$$

Hint: Put $A = \sqrt{a^2 + b^2}$ and choose $0 < \theta < \pi/2$ with $\cos \theta = a/A$ and $\sin \theta = b/A$. Integrate e^{-Az} around the sector of radius R and angle θ .

- (2) **Star-shaped open sets.** A nonempty open set $U \subseteq \mathbb{C}$ is *star-shaped with respect to* $a \in U$ if the segment $[a, z]$ lies in U for every $z \in U$.

(a) Prove that U is path connected.

(b) For a closed continuous curve $\gamma: [0, 1] \rightarrow U$ based at a , construct a homotopy with fixed endpoints from γ to the constant curve at a . Deduce that U is simply connected.

For the last deduction, you may use the elementary fact that in a path connected set it suffices to contract loops at one fixed base point. A connecting path and its reverse transfer the contraction to any other base point.

- (3) **Images of simply connected sets.** Let U be a simply connected open set and let f be a nonconstant holomorphic function on U . Must $f(U)$ be simply connected? Investigate this using

$$H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}, \quad f(z) = e^{2\pi iz}.$$

Determine $f(H)$ exactly and justify your conclusion about its simple connectivity, for example by integrating $1/z$ around a suitable circle.

- (4) **Two branches of the logarithm.** Let L_+ be the logarithm on $\mathbb{C} \setminus [0, \infty)$ with argument in $(0, 2\pi)$, and let L_- be the logarithm on $\mathbb{C} \setminus (-\infty, 0]$ with argument in $(-\pi, \pi)$. For *each* branch L , compute

$$L(-1+i), \quad \exp(iL(-i)), \quad L(-1-i).$$

Here $(-i)^i$ is defined relative to the chosen branch as $\exp(iL(-i))$. Explain how L_+ and L_- differ on the upper and lower half-planes.

- (5) **Holomorphic roots and the choice of domain.** Fix an integer $n \geq 2$.

(a) Let $U \subseteq \mathbb{C} \setminus \{0\}$ be a simply connected open set. Use a holomorphic logarithm on U to construct a holomorphic function g satisfying $g(z)^n = z$ on U .

(b) For $\theta \in \mathbb{R}$, give such a branch on the slit plane

$$U_\theta = \mathbb{C} \setminus \{re^{i\theta} : r \geq 0\},$$

specifying the interval used for the argument.

(c) Show that there is no continuous function g on $\mathbb{C} \setminus \{0\}$ satisfying $g(z)^n = z$ everywhere.
Hint: On the unit circle, consider $g(e^{it})e^{-it/n}$ for $0 \leq t \leq 2\pi$.

- (6) **Polynomial growth and Cauchy estimates.** Let f be entire and put

$$M_f(R) = \max_{|z|=R} |f(z)|.$$

Suppose there are a constant $C > 0$, an integer $k \geq 0$, and radii $R_j \rightarrow \infty$ such that $M_f(R_j) \leq CR_j^k$ for every j . Use Cauchy's estimates for the Taylor coefficients at 0 to prove that f is a polynomial of degree at most k (including the zero polynomial). What does the case $k = 0$ say?

- (7) **Laurent expansions.**

(a) Find a series in integer powers of z representing

$$f(z) = \frac{1}{(z-1)(z-2)}$$

in each of the regions $|z| < 1$, $1 < |z| < 2$, and $|z| > 2$. Justify convergence in each region.

(b) Find the first four nonzero terms of the Laurent expansion of

$$\frac{e^z}{z(1+z^2)}$$

valid for $0 < |z| < 1$.