

PROBLEM SET 2: SOLUTIONS
MATH 411: HONORS COMPLEX VARIABLES, FALL 2026

(1) Give the terms of order ≤ 3 in the power series (using the expressions for $\cos(T)$, $\sin(T)$, and e^T given in lecture):

- (a) $\frac{\sin T}{\cos T}$;
 (b) $\frac{e^T}{\sin T}$.

Solution. In class we defined

$$\begin{aligned}\cos T &= 1 - \frac{T^2}{2!} + \frac{T^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{T^{2n}}{(2n)!}, \\ \sin T &= T - \frac{T^3}{3!} + \frac{T^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{T^{2n+1}}{(2n+1)!}, \\ e^T &= 1 + T + \frac{T^2}{2!} + \frac{T^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{T^n}{n!}.\end{aligned}$$

(a) Write

$$\frac{1}{\cos T} = b_0 + b_1 T + b_2 T^2 + \cdots,$$

so that

$$(b_0 + b_1 T + b_2 T^2 + \cdots) \left(1 - \frac{T^2}{2!} + \frac{T^4}{4!} - \cdots \right) = 1.$$

Comparing coefficients:

$$b_0 = 1 \quad (\text{coefficient of } T^0),$$

$$b_1 = 0 \quad (\text{coefficient of } T),$$

$$b_2 - \frac{b_0}{2} = 0, \text{ so } b_2 = \frac{1}{2} \quad (\text{coefficient of } T^2).$$

Hence

$$\frac{1}{\cos T} = 1 + \frac{1}{2} T^2 + \cdots$$

and

$$\begin{aligned}\frac{\sin T}{\cos T} &= \left(1 + \frac{1}{2} T^2 + \cdots \right) \left(T - \frac{T^3}{3!} + \frac{T^5}{5!} - \cdots \right) \\ &= T + \left(\frac{1}{2} - \frac{1}{6} \right) T^3 + \cdots = \boxed{T + \frac{T^3}{3}} + \cdots.\end{aligned}$$

(b) Write

$$\sin T = T \left(1 - \frac{T^2}{3!} + \frac{T^4}{5!} - \cdots \right) = T(1 - h(T)),$$

where

$$h(T) = \frac{T^2}{3!} - \frac{T^4}{5!} + \cdots.$$

Since h has no constant term,

$$\begin{aligned} \frac{1}{\sin T} &= \frac{1}{T} (1 + h(T) + h(T)^2 + \cdots) = \frac{1}{T} \left(1 + \frac{T^2}{3!} + \left(\frac{1}{36} - \frac{1}{120} \right) T^4 + \cdots \right) \\ &= \frac{1}{T} \left(1 + \frac{T^2}{3!} + \frac{7T^4}{360} + \cdots \right). \end{aligned}$$

So

$$\begin{aligned} \frac{e^T}{\sin T} &= \frac{1}{T} \left(1 + T + \frac{T^2}{2!} + \frac{T^3}{3!} + \frac{T^4}{4!} + \cdots \right) \\ &\quad \cdot \left(1 + \frac{T^2}{3!} + \frac{7T^4}{360} + \cdots \right) \\ &= \frac{1}{T} \left(1 + T + \frac{2}{3}T^2 + \frac{1}{3}T^3 + \frac{13}{90}T^4 + \cdots \right) \\ &= \boxed{\frac{1}{T} + 1 + \frac{2T}{3} + \frac{T^2}{3} + \frac{13T^3}{90}} + \cdots. \end{aligned}$$

(For the T^4 coefficient inside the parentheses: $\frac{1}{24} + \frac{1}{12} + \frac{7}{360} = \frac{52}{360} = \frac{13}{90}$.) □

(2) Given complex numbers a_0, a_1, u_1, u_2 , define a_n for $n \geq 2$ by

$$a_n = u_1 a_{n-1} + u_2 a_{n-2}.$$

If

$$T^2 - u_1 T - u_2 = (T - \alpha)(T - \alpha')$$

with $\alpha \neq \alpha'$, show that

$$a_n = A\alpha^n + B(\alpha')^n$$

for suitable A, B . Find A, B in terms of $a_0, a_1, \alpha, \alpha'$. Consider the power series

$$F(T) = \sum_{n=0}^{\infty} a_n T^n.$$

Show that it represents a rational function $f(T)/g(T)$, where $f(T)$ and $g(T)$ are polynomials with $g(T) \neq 0$. Give the partial fraction decomposition of this rational function.

Solution. If $T^2 - u_1 T - u_2 = (T - \alpha)(T - \alpha')$, then we first note that

$$u_1 = \alpha + \alpha' \quad \text{and} \quad u_2 = -\alpha\alpha'. \tag{1.1}$$

Now, if $a_n = A\alpha^n + B(\alpha')^n$, we must have in particular that

$$a_1 = A\alpha + B\alpha' \quad \text{and} \quad a_0 = A + B. \tag{1.2}$$

These can be solved for A, B to yield

$$A = \frac{a_1 - a_0\alpha'}{\alpha - \alpha'} \quad \text{and} \quad B = a_0 - A = \frac{a_0\alpha - a_1}{\alpha - \alpha'}. \quad (1.3)$$

Note that the ratio involved in the definition of A is well-defined as we assumed $\alpha \neq \alpha'$. So let us now *define* A and B as in (1.3) and verify that with these definitions we must have

$$a_n = A\alpha^n + B(\alpha')^n \quad \text{for any } n \geq 0. \quad (1.4)$$

Indeed, for $n = 0, 1$ this follows immediately from the way we have defined A and B , so let us assume the formula (1.4) holds for a_n when $n = 0, \dots, m$ with $m \geq 1$. We argue that it must also hold for $m+1$, i.e. $a_{m+1} = A\alpha^{m+1} + B(\alpha')^{m+1}$. Indeed, by hypothesis, we have

$$\begin{aligned} a_{m+1} &= u_1 a_m + u_2 a_{m-1} = u_1 (A\alpha^m + B(\alpha')^m) + u_2 (A\alpha^{m-1} + B(\alpha')^{m-1}) \\ &= A\alpha^{m-1}(u_1\alpha + u_2) + B(\alpha')^{m-1}(u_1\alpha' + u_2). \end{aligned}$$

Using (1.1) (equivalently, since α, α' are roots of $T^2 - u_1T - u_2$), we have $u_1\alpha + u_2 = \alpha^2$ and $u_1\alpha' + u_2 = (\alpha')^2$, so the induction is proven.

Defining $F(T)$ as in the problem, we find

$$\begin{aligned} u_1TF(T) &= \sum_{n \geq 1} u_1 a_{n-1} T^n = u_1 a_0 T + u_1 a_1 T^2 + \dots, \\ u_2 T^2 F(T) &= \sum_{n \geq 2} u_2 a_{n-2} T^n = u_2 a_0 T^2 + u_2 a_1 T^3 + \dots, \end{aligned}$$

and therefore, using the recursion for the coefficients of T^n with $n \geq 2$, we have (formally)

$$F(T) - a_0 - a_1 T = u_1 T F(T) - u_1 a_0 T + u_2 T^2 F(T).$$

Solving for $F(T)$ we find

$$F(T) = \frac{a_0 + a_1 T - u_1 a_0 T}{1 - u_1 T - u_2 T^2}, \quad (1.5)$$

so $F = f/g$ with $f(T) = a_0 + (a_1 - u_1 a_0)T$ and $g(T) = 1 - u_1 T - u_2 T^2$; note $g \neq 0$ since its constant term is 1. From (1.1) we find that

$$(1 - \alpha T)(1 - \alpha' T) = 1 - (\alpha + \alpha')T + \alpha\alpha' T^2 = 1 - u_1 T - u_2 T^2,$$

so that

$$F(T) = \frac{a_0 + a_1 T - u_1 a_0 T}{(1 - \alpha T)(1 - \alpha' T)} = \boxed{\frac{A}{1 - \alpha T} + \frac{B}{1 - \alpha' T}},$$

with A, B as in (1.3). This last equality follows immediately from clearing denominators and using (1.1) and (1.2): indeed $A(1 - \alpha' T) + B(1 - \alpha T) = (A + B) - (A\alpha' + B\alpha)T$, and $A\alpha' + B\alpha = (A + B)(\alpha + \alpha') - (A\alpha + B\alpha') = u_1 a_0 - a_1$. (Alternatively, expand each term as a geometric series, $\frac{A}{1 - \alpha T} = \sum A\alpha^n T^n$, and compare with (1.4).) $\square$

(3) Determine the radius of convergence of the following series:

(a) $\sum n^2 z^n$;

(b) $\sum \frac{n!}{n} z^n$;

- (c) If $f = \sum a_n z^n$ has radius of convergence $r > 0$, show that $\sum n^d a_n z^n$ has the same radius of convergence for any positive integer d .

Solution. (a) It is $\boxed{1}$. We use Hadamard's criterion and compute

$$\limsup_{n \rightarrow \infty} |n^2|^{1/n} = \limsup_{n \rightarrow \infty} n^{2/n} = \limsup_{n \rightarrow \infty} e^{\frac{2 \log n}{n}} = 1,$$

since $\lim_{n \rightarrow \infty} \frac{2 \log n}{n} = 0$.

(b) First note that

$$\lim_{n \rightarrow \infty} (1/n)^{1/n} = 1.$$

Also, as we observed in class, we have the Stirling estimate $n! = n^n e^{-n} u_n$ where $\lim_{n \rightarrow \infty} u_n^{1/n} = 1$. Thus $(n!)^{1/n} = n e^{-1} u_n^{1/n}$, and

$$\limsup_{n \rightarrow \infty} (n!)^{1/n} (1/n)^{1/n} = \limsup_{n \rightarrow \infty} n e^{-1} u_n^{1/n} (1/n)^{1/n} = \infty,$$

so the radius of convergence is $\boxed{0}$.

(c) Again, we use Hadamard's criterion to compute $\limsup_{n \rightarrow \infty} |n^d a_n|^{1/n}$. First,

$$\lim_{n \rightarrow \infty} n^{d/n} = \lim_{n \rightarrow \infty} e^{\frac{d \log n}{n}} = 1.$$

Since $|n^d a_n|^{1/n} = n^{d/n} |a_n|^{1/n}$, and $n^{d/n}$ is a sequence of positive numbers with limit 1, the remarks before Theorem 2.6 in Lang (p. 55) (if $\lim t_n = t$ exists and $t \neq 0, \infty$, then $\limsup t_n s_n = t \limsup s_n$) give

$$\limsup_{n \rightarrow \infty} |n^d a_n|^{1/n} = \limsup_{n \rightarrow \infty} |a_n|^{1/n} = 1/r.$$

Hence $\sum n^d a_n z^n$ also has radius of convergence r . □

(4) Prove the following:

- (a) The power series $\sum n z^n$ has radius of convergence 1 but does not converge at any point of the unit circle.
- (b) The power series $\sum \frac{z^n}{n^2}$ has radius of convergence 1 and converges at every point of the unit circle.
- (c) The power series $\sum \frac{z^n}{n}$ converges at every point of the unit circle except $z = 1$.

Solution. (a) Since $\limsup_{n \rightarrow \infty} n^{1/n} = 1$, the assertion about the radius of convergence being equal to 1 follows from Hadamard's theorem. On the other hand, if $|z| = 1$, then the terms $n z^n$ of the series have norm n , which approaches infinity, so the series clearly cannot converge for any such z .

(b) Since $\limsup_{n \rightarrow \infty} \frac{1}{n^{2/n}} = 1$, the radius of convergence is 1. If $|z| = 1$, then

$$\sum \left| \frac{z^n}{n^2} \right| = \sum \frac{1}{n^2},$$

and the latter converges by comparison with (a Riemann sum for) the convergent integral $\int_1^\infty \frac{1}{x^2} dx$, for example. So the series converges absolutely at every point of the unit circle.

(c) Again, from Hadamard we see the radius of convergence is 1. For $z = 1$, this is the harmonic series, whose divergence is well known. So let us assume that $z \neq 1$ but z lies on the unit circle.

We use *summation by parts*: if $\{a_n\}_{n=1}^N$ and $\{b_n\}_{n=1}^N$ are finite sequences of complex numbers and $B_k = \sum_{n=1}^k b_n$, then

$$\sum_{n=1}^N a_n b_n = a_N B_N - \sum_{n=1}^{N-1} (a_{n+1} - a_n) B_n. \quad (1.6)$$

(Proof by induction on N : for $N = 1$ both sides are $a_1 b_1$. If L_N, R_N denote the left and right hand sides, then $L_{N+1} - L_N = a_{N+1} b_{N+1}$, while $R_{N+1} - R_N = a_{N+1} B_{N+1} - a_N B_N - (a_{N+1} - a_N) B_N = a_{N+1} (B_{N+1} - B_N) = a_{N+1} b_{N+1}$.)

Apply (1.6) to $\sum_{n=1}^N \frac{1}{n} z^n$ with $a_n = 1/n$ and $b_n = z^n$. As $z \neq 1$ we may write $B_n = \frac{z - z^{n+1}}{1 - z}$, and since $a_{n+1} - a_n = -\frac{1}{n(n+1)}$, we obtain

$$\sum_{n=1}^N \frac{1}{n} z^n = \frac{1}{1 - z} \cdot \frac{z - z^{N+1}}{N} + \frac{1}{1 - z} \sum_{n=1}^{N-1} \frac{z - z^{n+1}}{n(n+1)}.$$

Now if z is on the unit circle we have $|z - z^k| \leq 2$ for any integer k . Hence the first term is bounded in norm by $\frac{2}{|1 - z|N}$, which tends to 0 as $N \rightarrow \infty$. In the second term, the n -th summand of the sum is bounded in norm by $\frac{2}{n(n+1)}$, and $\sum \frac{2}{n(n+1)}$ converges (it telescopes: $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$). So the series $\sum_{n \geq 1} \frac{z - z^{n+1}}{n(n+1)}$ converges absolutely. Letting $N \rightarrow \infty$, the partial sums $\sum_{n=1}^N \frac{z^n}{n}$ converge, i.e. $\sum \frac{z^n}{n}$ converges at every $z \neq 1$ with $|z| = 1$. $\square$

(5) Find the terms of order ≤ 3 in the power series expansion of

$$f(z) = \frac{z - 2}{(z + 3)(z + 2)}$$

at $z = 1$.

Solution. We begin by noting that if $|z/a| < 1$ we have

$$\frac{1}{a + z} = \frac{1}{a} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{a}\right)^n.$$

Note that

$$z - 2 = -1 + (z - 1), \quad z + 3 = 4 + (z - 1), \quad z + 2 = 3 + (z - 1).$$

Hence

$$\begin{aligned} \frac{1}{z + 3} &= \frac{1}{4 + (z - 1)} = \frac{1}{4} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z - 1}{4}\right)^n, \\ \frac{1}{z + 2} &= \frac{1}{3 + (z - 1)} = \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z - 1}{3}\right)^n, \end{aligned}$$

and so, for $|z - 1| < 3$,

$$\begin{aligned}
 \frac{z-2}{(z+3)(z+2)} &= \frac{1}{12}(-1 + (z-1)) \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-1}{4}\right)^n \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-1}{3}\right)^n \\
 &= \frac{1}{12}(-1 + (z-1)) \\
 &\quad \cdot \left(1 - \frac{7}{12}(z-1) + \frac{37}{144}(z-1)^2 - \frac{175}{1728}(z-1)^3 + \dots\right) \\
 &= \boxed{-\frac{1}{12} + \frac{19}{144}(z-1) - \frac{121}{1728}(z-1)^2 + \frac{619}{20736}(z-1)^3} + \dots
 \end{aligned}$$

in a neighbourhood of $z = 1$. (Here the coefficient of $(z-1)^n$ in the product of the two geometric series is $(-1)^n \sum_{k=0}^n 4^{-k} 3^{-(n-k)}$; e.g. for $n = 2$ it is $\frac{1}{16} + \frac{1}{12} + \frac{1}{9} = \frac{37}{144}$.) $\square$