

PROBLEM SET 1
MATH 411: HONORS COMPLEX VARIABLES, FALL 2026

- (1) (a) Put $-1 - i$ into polar form
 (b) Put $e^{-5i\pi/4}$ into the form $x + iy$.
 (c) Find the real and imaginary parts of $i^{1/4}$, taking the fourth root whose angle lies between 0 and $\pi/2$.
 (d) Describe geometrically the set of points $z \in \mathbb{C}$ such that $|z| = \operatorname{Re}(z) + 1$.
- (2) Let $f(z) = e^z$. Describe the image under f of the following sets
 (a) The set of $z = x + iy$ such that $x \leq 1$ and $0 \leq y \leq \pi$
 (b) The set of $z = x + iy$ such that $0 \leq y \leq \pi$ (no condition on x)
- (3) Let f be the function defined by

$$f(z) = \lim_{n \rightarrow \infty} \frac{1}{1 + n^2 z}.$$

Show that f is the characteristic function of the set $\{0\}$, i.e. $f(0) = 1$ and $f(z) = 0$ otherwise.

- (4) (a) Let z, w be two complex numbers such that $\bar{z}w \neq 1$. Prove that

$$\left| \frac{w - z}{1 - \bar{w}z} \right| < 1 \text{ if } |z| < 1 \text{ and } |w| < 1$$

and also that

$$\left| \frac{w - z}{1 - \bar{w}z} \right| = 1 \text{ if } |z| = 1 \text{ or } |w| = 1$$

Hint: reduce to the case where z is real, say $z = r$, in which case it suffices to prove that

$$(r - w)(r - \bar{w}) \leq (1 - rw)(1 - r\bar{w})$$

with equality when $|r| = 1$ or $|w| = 1$.

- (b) Prove that for a fixed w in the unit disc $\mathbb{D}$ the mapping

$$F : z \mapsto \frac{w - z}{1 - \bar{w}z}$$

satisfies the following conditions:

- (i) F maps $\mathbb{D}$ to itself and is holomorphic on $\mathbb{D}$
 - (ii) F interchanges 0 and w (i.e. $F(0) = w$ and $F(w) = 0$)
 - (iii) $|F(z)| = 1$ if $|z| = 1$
 - (iv) $F : \mathbb{D} \rightarrow \mathbb{D}$ is bijective (Hint: calculate $F \circ F$).
- (5) Show that in polar coordinates, i.e. if $F(x, y)$ is regarded as $F(r, \theta) = (u(r, \theta), v(r, \theta))$, the Cauchy–Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \text{ and } \frac{1}{r} \frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Use these equations to show that the logarithm function defined as

$$\log(z) := \log(r) + i\theta \text{ where } z = re^{i\theta} \text{ with } -\pi < \theta < \pi$$

is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.

(6) (a) Show that

$$4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} = 4 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} = \Delta$$

where Δ is the Laplacian,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

(b) Show that if f is holomorphic on an open set $U \subset \mathbb{C}$, then $\psi(x, y) = \operatorname{Re}(f)$ and $\phi(x, y) = \operatorname{Im}(f)$ are both harmonic functions (i.e. they satisfy $\Delta\psi = \Delta\phi = 0$).

(7) Show that $f(x + iy) = \sqrt{|x||y|}$ satisfies the Cauchy-Riemann equations at the origin but that f is not holomorphic at 0

(8) Suppose that f is holomorphic in an open disc $\mathcal{D} \subset \mathbb{C}$, say $f = u(x, y) + iv(x, y)$. Prove that

(a) if $|f|^2 = u^2 + v^2$ is constant, then f is constant;

(b) if there exist real, non-zero constants a, b , and c such that

$$au(x, y) + bv(x, y) = c,$$

then f is constant.