

PROBLEM SET 1: SOLUTIONS
MATH 411: HONORS COMPLEX VARIABLES, FALL 2026

- (1) (a) Put $-1 - i$ into polar form

Solution. $|-1 - i| = \sqrt{2}$, and an argument is $5\pi/4$ (or $-3\pi/4$). Hence

$$-1 - i = \sqrt{2}e^{5\pi i/4} = \sqrt{2}e^{-3\pi i/4}.$$

□

- (b) Put $e^{-5\pi i/4}$ into the form $x + iy$.

Solution. By Euler's formula,

$$e^{-5\pi i/4} = \cos(-5\pi/4) + i\sin(-5\pi/4) = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}.$$

□

- (c) Find the real and imaginary parts of $i^{1/4}$, taking the fourth root whose angle lies between 0 and $\pi/2$.

Solution. The required root is $e^{i\pi/8}$, so

$$i^{1/4} = \frac{\sqrt{2+\sqrt{2}}}{2} + i\frac{\sqrt{2-\sqrt{2}}}{2}.$$

Thus its real and imaginary parts are $\sqrt{2+\sqrt{2}}/2$ and $\sqrt{2-\sqrt{2}}/2$, respectively. □

- (d) Describe geometrically the set of points $z \in \mathbb{C}$ such that $|z| = \operatorname{Re}(z) + 1$.

Solution. Writing $z = x + iy$ and squaring gives

$$x^2 + y^2 = (x+1)^2 \iff y^2 = 2x+1.$$

This is the parabola $x = (y^2 - 1)/2$, with vertex $(-1/2, 0)$ and opening to the right. Its points have $x+1 \geq 1/2$, so squaring added no points. □

- (2) Let $f(z) = e^z$. Describe the image under f of the following sets

- (a) The set of $z = x + iy$ such that $x \leq 1$ and $0 \leq y \leq \pi$

Solution. Since $e^{x+iy} = e^x e^{iy}$, the modulus ranges over $(0, e]$ and the argument over $[0, \pi]$. The image is

$$\{\zeta \in \mathbb{C} : 0 < |\zeta| \leq e, \operatorname{Im}\zeta \geq 0\},$$

the closed upper half-disc of radius e , with the origin removed. □

- (b) The set of $z = x + iy$ such that $0 \leq y \leq \pi$ (no condition on x)

Solution. Here the modulus ranges over all positive numbers, so the image is

$$\{\zeta \in \mathbb{C} : \zeta \neq 0, \operatorname{Im}\zeta \geq 0\},$$

the closed upper half-plane with the origin removed. □

(3) Let f be the function defined by

$$f(z) = \lim_{n \rightarrow \infty} \frac{1}{1 + n^2 z}.$$

Show that f is the characteristic function of the set $\{0\}$, i.e. $f(0) = 1$ and $f(z) = 0$ otherwise.

Solution. For $z = 0$, every term is 1. If $z \neq 0$, then, once $n^2|z| > 1$,

$$\left| \frac{1}{1 + n^2 z} \right| \leq \frac{1}{n^2|z| - 1} \rightarrow 0.$$

Therefore $f(0) = 1$ and $f(z) = 0$ for every $z \neq 0$. □

(4) (a) Let z, w be two complex numbers such that $\bar{z}w \neq 1$. Prove that

$$\left| \frac{w - z}{1 - \bar{w}z} \right| < 1 \text{ if } |z| < 1 \text{ and } |w| < 1$$

and also that

$$\left| \frac{w - z}{1 - \bar{w}z} \right| = 1 \text{ if } |z| = 1 \text{ or } |w| = 1$$

Hint: reduce to the case where z is real, say $z = r$, in which case it suffices to prove that

$$(r - w)(r - \bar{w}) \leq (1 - rw)(1 - r\bar{w})$$

with equality when $|r| = 1$ or $|w| = 1$.

Solution. Expanding both sides gives

$$|1 - \bar{w}z|^2 - |w - z|^2 = (1 - |w|^2)(1 - |z|^2).$$

If $|z| < 1$ and $|w| < 1$, the right side is positive, and division by $|1 - \bar{w}z|^2$ gives the strict inequality. If $|z| = 1$ or $|w| = 1$, the right side is zero, so the quotient has modulus 1. □

(b) Prove that for a fixed w in the unit disc $\mathbb{D}$ the mapping

$$F : z \mapsto \frac{w - z}{1 - \bar{w}z}$$

satisfies the following conditions:

- (i) F maps $\mathbb{D}$ to itself and is holomorphic on $\mathbb{D}$
- (ii) F interchanges 0 and w (i.e. $F(0) = w$ and $F(w) = 0$)
- (iii) $|F(z)| = 1$ if $|z| = 1$
- (iv) $F : \mathbb{D} \rightarrow \mathbb{D}$ is bijective (Hint: calculate $F \circ F$).

Solution. Since $|\bar{w}z| < 1$ for $z, w \in \mathbb{D}$, the denominator never vanishes in $\mathbb{D}$; hence F is holomorphic there. The preceding part shows that $|F(z)| < 1$, so $F(\mathbb{D}) \subseteq \mathbb{D}$. Also

$$F(0) = w, \quad F(w) = 0,$$

and the preceding part gives $|F(z)| = 1$ when $|z| = 1$.

Finally, direct substitution yields

$$F(F(z)) = \frac{w - \frac{w - z}{1 - \bar{w}z}}{1 - \bar{w} \frac{w - z}{1 - \bar{w}z}} = z.$$

Thus F is its own inverse and is therefore a bijection of $\mathbb{D}$. □

- (5) Show that in polar coordinates, i.e. if $F(x, y)$ is regarded as $F(r, \theta) = (u(r, \theta), v(r, \theta))$, the Cauchy–Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \text{ and } \frac{1}{r} \frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Use these equations to show that the logarithm function defined as

$$\log(z) := \log(r) + i\theta \text{ where } z = re^{i\theta} \text{ with } -\pi < \theta < \pi$$

is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.

Solution. The chain rule gives

$$u_r = u_x \cos \theta + u_y \sin \theta, \quad u_\theta = -ru_x \sin \theta + ru_y \cos \theta,$$

and the analogous formulas for v . Substituting $u_x = v_y$ and $u_y = -v_x$ gives

$$u_r = \frac{1}{r} v_\theta, \quad \frac{1}{r} u_\theta = -v_r.$$

For the given logarithm, $u = \log r$ and $v = \theta$. Thus $u_r = 1/r$, $u_\theta = 0$, $v_r = 0$, and $v_\theta = 1$, so both polar Cauchy–Riemann equations hold. The partial derivatives are continuous for $r > 0$ and $-\pi < \theta < \pi$, so this branch is holomorphic there. $\square$

- (6) (a) Show that

$$4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} = 4 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} = \Delta$$

where Δ is the Laplacian,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Solution. Using

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y),$$

and the equality of mixed partials, we obtain

$$4\partial_z\partial_{\bar{z}} = (\partial_x - i\partial_y)(\partial_x + i\partial_y) = \partial_x^2 + \partial_y^2 = \Delta.$$

Reversing the order gives the same calculation. $\square$

- (b) Show that if f is holomorphic on an open set $U \subset \mathbb{C}$, then $\psi(x, y) = \operatorname{Re}(f)$ and $\phi(x, y) = \operatorname{Im}(f)$ are both harmonic functions (i.e. they satisfy $\Delta\psi = \Delta\phi = 0$).

Solution. The Cauchy–Riemann equations are $\psi_x = \phi_y$ and $\psi_y = -\phi_x$. Since holomorphic functions are smooth,

$$\Delta\psi = \psi_{xx} + \psi_{yy} = \phi_{yx} - \phi_{xy} = 0, \quad \Delta\phi = \phi_{xx} + \phi_{yy} = -\psi_{yx} + \psi_{xy} = 0.$$

Thus both real and imaginary parts are harmonic. $\square$

- (7) Show that $f(x + iy) = \sqrt{|x||y|}$ satisfies the Cauchy–Riemann equations at the origin but that f is not holomorphic at 0

Solution. Write $f = u + iv$, where $u(x, y) = \sqrt{|x||y|}$ and $v = 0$. On either coordinate axis u is zero, so

$$u_x(0, 0) = u_y(0, 0) = v_x(0, 0) = v_y(0, 0) = 0.$$

Thus the Cauchy–Riemann equations hold at the origin. But along the real axis the difference quotient is zero, whereas for $t > 0$,

$$\frac{f(t+it) - f(0)}{t+it} = \frac{t}{t(1+i)} = \frac{1}{1+i}.$$

The limits disagree, so f is not complex differentiable at 0 and hence is not holomorphic there. $\square$

- (8) Suppose that f is holomorphic in an open disc $\mathcal{D} \subset \mathbb{C}$, say $f = u(x, y) + iv(x, y)$. Prove that
(a) if $|f|^2 = u^2 + v^2$ is constant, then f is constant;

Solution. Suppose $|f|^2 = c$. If $c = 0$ we have nothing to show, so assume $c = u^2 + v^2 > 0$. Differentiating the expression $|f|^2 = c$ yields

$$2uu_x + 2vv_x = 0,$$

$$2uu_y + 2vv_y = 0.$$

Applying Cauchy–Riemann (i.e. $u_x = v_y$, $u_y = -v_x$), we find

$$2uu_x - 2vu_y = 0,$$

$$2uu_y + 2vu_x = 0.$$

Since

$$\det \begin{pmatrix} u & -v \\ v & u \end{pmatrix} = u^2 + v^2 \neq 0,$$

the only solution to this system is $u_x = u_y = 0$. In a similar way we also conclude that $v_x = v_y = 0$. As the Jacobian of the map $F = (u(x, y), v(x, y))$ vanishes, the map F is locally constant, and hence constant since $\mathcal{D}$ is connected. $\square$

- (b) if there exist real, non-zero constants a, b , and c such that

$$au(x, y) + bv(x, y) = c,$$

then f is constant.

Solution. Differentiating the above expression and applying Cauchy–Riemann yields the system

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix} \begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

As

$$\det \begin{pmatrix} a & b \\ b & -a \end{pmatrix} = -(a^2 + b^2) \neq 0,$$

we must have $u_x = v_x = 0$. Again applying Cauchy–Riemann, we also conclude that $u_y = v_y = 0$, and we argue as above. $\square$