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1. DIHEDRAL GROUPS

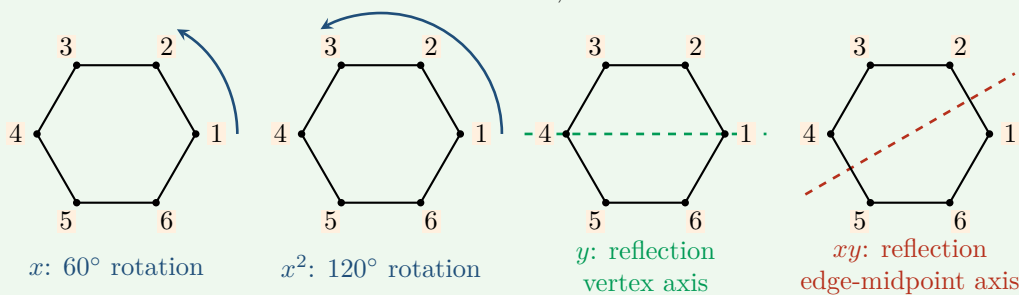
Definition 1.1 (Dihedral group). For an integer $n \geq 3$, the *dihedral group* D_{2n} is the group generated by x and y with relations

$$x^n = 1, \quad y^2 = 1, \quad yxy^{-1} = x^{-1}.$$

The last relation is equivalent to $yx = x^{-1}y = x^{n-1}y$. One can verify that the relation $yx = x^{-1}y$ allows one to write every element as either x^i or x^iy , where $0 \leq i < n$. So D_{2n} consists of $2n$ elements:

$$\{1, x, x^2, \dots, x^{n-1}, y, xy, x^2y, \dots, x^{n-1}y\}.$$

Example 1.1. Last time we dealt with $n = 4$ and the square. Consider now the regular hexagon, $n = 6$ and D_{12} . It has 12 elements. What are they? Label the vertices counterclockwise as shown below. The blue arrows show two rotations; the dashed lines show two reflection axes.



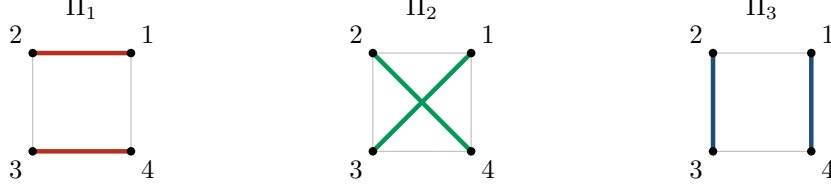
In the homework, you will work out this example in more detail.

2. BACK TO CORRESPONDENCE THEOREM EXAMPLE

Recall our surjective homomorphism $\varphi : S_4 \rightarrow S_3$ induced by the action of S_4 on the three pairings

$$\Pi_1 = \{\{1, 2\}, \{3, 4\}\}, \quad \Pi_2 = \{\{1, 3\}, \{2, 4\}\}, \quad \Pi_3 = \{\{1, 4\}, \{2, 3\}\}.$$

These are the three configurations below. The colored segments indicate the two pairs in each configuration.



The kernel of $\varphi : S_4 \rightarrow S_3$ is the Klein four group consisting of the following four elements:

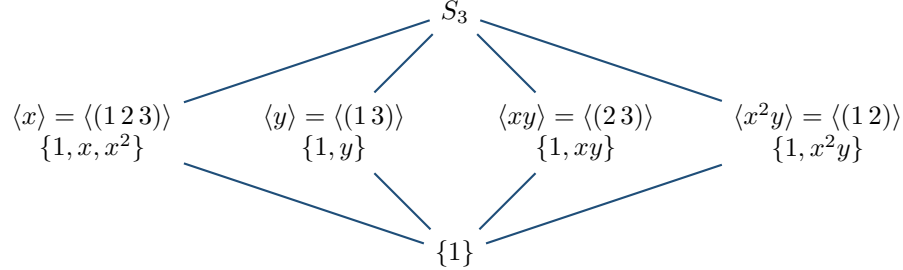
$$K = \{1, (12)(34), (13)(24), (14)(23)\}.$$

Write $x_1 = (1234)$ and $r_1 = (24)$ for the square symmetries.

Recall that we have written $S_3 = \{1, x, x^2, y, xy, x^2y\}$, where $x^3 = y^2 = 1$ and $xyx^{-1} = x^{-1}$. Concretely, take $x = (123)$ and $y = (13)$. Then $x^2 = (132)$, $xy = (23)$, and $x^2y = (12)$, with permutation products composed from right to left. Its subgroups are

$$\begin{aligned} \{1\}, \quad \langle x \rangle = \langle (123) \rangle, \quad \langle y \rangle = \langle (13) \rangle, \\ \langle xy \rangle = \langle (23) \rangle, \quad \langle x^2y \rangle = \langle (12) \rangle, \quad S_3. \end{aligned}$$

The subgroup inclusion graph is shown below; inclusions run upward.



Consequently, there are exactly six subgroups of S_4 containing K . Their orders follow from $|\varphi^{-1}(\mathcal{H})| = 4|\mathcal{H}|$:

$\mathcal{H} \leq S_3$	$\varphi^{-1}(\mathcal{H}) \leq S_4$	$ \varphi^{-1}(\mathcal{H}) $
$\{1\}$	K	4
$\langle x \rangle = \langle (123) \rangle$	A	12
$\langle y \rangle = \langle (13) \rangle$	H_1	8
$\langle xy \rangle = \langle (23) \rangle$	H_2	8
$\langle x^2y \rangle = \langle (12) \rangle$	H_3	8
S_3	S_4	24

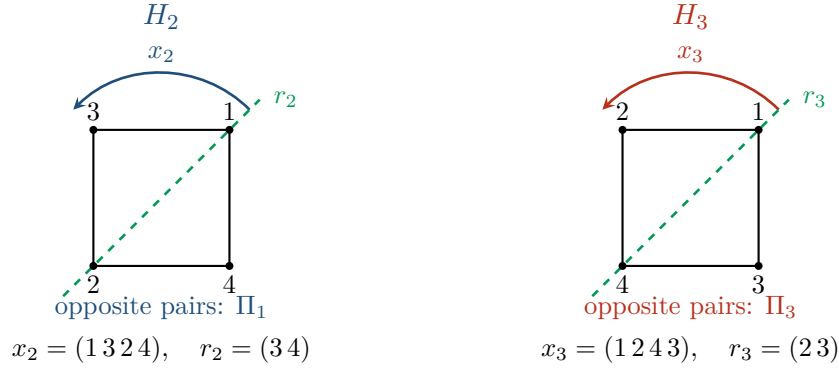
What are the groups A, H_1, H_2, H_3 ? we already identified $A = A_4$. Let's now work with

$$H_1 = \varphi^{-1}(\{1, (13)\}).$$

We know that $|H_1| = 8$. Any guesses as to what it is?

Note that: $\varphi(x_1) = \varphi(r_1) = (13) = y$, so $\langle x_1, r_1 \rangle \leq \varphi^{-1}(\langle y \rangle)$. The preimage has order $|K| \cdot 2 = 8$ by the correspondence theorem, and hence $\langle x_1, r_1 \rangle = \varphi^{-1}(\langle y \rangle) = H_1$.

For H_2 and H_3 , relabel the vertices as shown below. The opposite vertex pairs now form Π_1 and Π_3 , respectively. The curved arrows indicate counterclockwise quarter-turns; the green dashed lines are diagonal reflection axes.



The induced permutations of the three pairings are

$$\varphi(x_2) = \varphi(r_2) = (2\ 3) = xy, \quad \varphi(x_3) = \varphi(r_3) = (1\ 2) = x^2y.$$

Each square symmetry group has order 8, as does the corresponding preimage. Therefore

$$H_2 = \langle x_2, r_2 \rangle = \varphi^{-1}(\langle xy \rangle),$$

$$H_3 = \langle x_3, r_3 \rangle = \varphi^{-1}(\langle x^2y \rangle).$$

Both are copies of D_8 inside S_4 containing K .

Exercise 2.1. Use the correspondence theorem to explain why none of the three subgroups H_1, H_2, H_3 is normal in S_4 .

3. EXTERNAL DIRECT PRODUCTS

There are two ways to approach products of groups:

- An *external* product starts with groups G_1, G_2 and constructs a new group $G_1 \times G_2$.
- An *internal* product starts with a group G and asks whether suitable subgroups $G_1, G_2 \leq G$ give an isomorphism $G \cong G_1 \times G_2$.

We first develop the external construction, then explain how to recognize a direct product inside a given group.

Definition 3.1 (External direct product). Let G_1 and G_2 be groups. Their *direct product* $G_1 \times G_2$ is the Cartesian product of their underlying sets, with multiplication defined componentwise:

$$(g_1, g_2)(g'_1, g'_2) = (g_1g'_1, g_2g'_2).$$

Lemma 3.1. The direct product is a group. Its identity is (e_1, e_2) and

$$(g_1, g_2)^{-1} = (g_1^{-1}, g_2^{-1}).$$

Exercise 3.1. Verify the group axioms for $G_1 \times G_2$, using the corresponding axioms in each factor. If both groups are finite, show that $|G_1 \times G_2| = |G_1||G_2|$.

3.1. The factors as normal subgroups. The maps

$$\begin{aligned} i_1 : G_1 &\longrightarrow G_1 \times G_2, & g_1 &\longmapsto (g_1, e_2), \\ i_2 : G_2 &\longrightarrow G_1 \times G_2, & g_2 &\longmapsto (e_1, g_2) \end{aligned}$$

are injective homomorphisms. We often identify the factors with their images, but the subgroups inside the product are, more precisely,

$$\tilde{G}_1 = G_1 \times \{e_2\}, \quad \tilde{G}_2 = \{e_1\} \times G_2.$$

Lemma 3.2. *Both $\tilde{G}_1$ and $\tilde{G}_2$ are normal in $G_1 \times G_2$, and*

$$\tilde{G}_1 \cap \tilde{G}_2 = \{(e_1, e_2)\}.$$

Proof. For $(g_1, g_2) \in G_1 \times G_2$ and $h_1 \in G_1$,

$$(g_1, g_2)(h_1, e_2)(g_1, g_2)^{-1} = (g_1 h_1 g_1^{-1}, e_2) \in \tilde{G}_1.$$

Thus $\tilde{G}_1$ is normal; the same computation in the second coordinate proves normality of $\tilde{G}_2$. An element in both subgroups must have first coordinate e_1 and second coordinate e_2 , proving the intersection statement. $\square$