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1. A WORD ON GROUP ACTIONS

Let S be a set and G a group. An action of G on S , or a group action, is defined as follows.

Definition 1.1 (Group action). A (left) action of G on S is a map

$$\circ : G \times S \longrightarrow S, \quad (g, s) \longmapsto g \circ s,$$

called the *action map*, such that:

- (1) $e_G \circ s = s$ for every $s \in S$.
- (2) $(gh) \circ s = g \circ (h \circ s)$ for all $g, h \in G$ and $s \in S$.

Let's give some examples which we have already seen before!

Example 1.1 (The natural action of S_n). The symmetric group S_n acts on $S = \{1, 2, \dots, n\}$ by evaluation:

$$S_n \times S \longrightarrow S, \quad (\sigma, i) \longmapsto \sigma \circ i := \sigma(i).$$

The identity permutation fixes every $i \in S$, and composition of permutations gives

$$(\sigma\tau) \circ i = (\sigma\tau)(i) = \sigma(\tau(i)) = \sigma \circ (\tau \circ i).$$

Thus both action axioms hold. For example, $(123) \in S_3$ sends 1 to 2, 2 to 3, and 3 to 1.

Example 1.2 (The natural action of $\text{GL}_n(\mathbb{R})$). The group $\text{GL}_n(\mathbb{R})$ of invertible $n \times n$ real matrices acts on $\mathbb{R}^n$ by matrix multiplication. Regarding vectors as column vectors, the action map is

$$\text{GL}_n(\mathbb{R}) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad (A, v) \longmapsto A \circ v := Av.$$

The identity matrix satisfies $I_n \circ v = I_n v = v$, and associativity of matrix multiplication gives

$$(AB) \circ v = (AB)v = A(Bv) = A \circ (B \circ v).$$

Thus both action axioms hold.

Example 1.3 (The action of S_4 on three partitions). Recall the three partitions from Lecture 16:

$$\Pi_1 = \{\{1, 2\}, \{3, 4\}\},$$

$$\Pi_2 = \{\{1, 3\}, \{2, 4\}\},$$

$$\Pi_3 = \{\{1, 4\}, \{2, 3\}\}.$$

Let $\mathcal{P} = \{\Pi_1, \Pi_2, \Pi_3\}$. The group S_4 acts on $\mathcal{P}$ by applying a permutation to every entry:

$$\sigma \circ \{\{a, b\}, \{c, d\}\} = \{\{\sigma(a), \sigma(b)\}, \{\sigma(c), \sigma(d)\}\}.$$

Since σ is a bijection, the result is again a partition into two unordered pairs, so it belongs to $\mathcal{P}$. The identity fixes each partition, and applying τ and then σ to every entry is the same as applying $\sigma\tau$. Therefore

$$(\sigma\tau) \circ \Pi_i = \sigma \circ (\tau \circ \Pi_i),$$

which verifies the second action axiom.

For example, if $\sigma = (1\ 2\ 3\ 4)$, then

$$\sigma \circ \Pi_1 = \Pi_3, \quad \sigma \circ \Pi_2 = \Pi_2, \quad \sigma \circ \Pi_3 = \Pi_1.$$

Identifying Π_i with i gives exactly the homomorphism $\varphi : S_4 \rightarrow S_3$ from Lecture 16: if $\sigma \circ \Pi_i = \Pi_j$, then $\varphi(\sigma)(i) = j$. In particular, $\varphi((1\ 2\ 3\ 4)) = (1\ 3)$.

1.1. Actions as homomorphisms. The previous example is actually a general fact.

Proposition 1.1. *An action of G on a set $S = \{s_1, \dots, s_d\}$ of cardinality d determines a homomorphism*

$$\rho : G \longrightarrow S_d.$$

The correspondence is given by $\rho(g)(i) = j$ if $g \circ s_i = s_j$.

Proof. We need to check that $\rho(g)$ is a permutation of $\{1, \dots, d\}$ and that the map $g \mapsto \rho(g)$ is a homomorphism, that is,

$$\rho(g_1 g_2) = \rho(g_1) \rho(g_2).$$

For the first part, the map $S \rightarrow S$ given by $s \mapsto g \circ s$ has inverse $s \mapsto g^{-1} \circ s$. Indeed, if $g \circ s_i = s_j$, then

$$g^{-1} \circ s_j = g^{-1} \circ (g \circ s_i) = (g^{-1}g) \circ s_i = s_i.$$

Likewise, $g \circ (g^{-1} \circ s_i) = s_i$. Thus $\rho(g)$ is a permutation of the indices, with inverse $\rho(g^{-1})$.

For the homomorphism property, suppose $\rho(g_2)(i) = j$ and $\rho(g_1)(j) = k$. Then $g_2 \circ s_i = s_j$ and $g_1 \circ s_j = s_k$, so

$$(g_1 g_2) \circ s_i = g_1 \circ (g_2 \circ s_i) = g_1 \circ s_j = s_k.$$

Consequently,

$$\rho(g_1 g_2)(i) = k = \rho(g_1)(\rho(g_2)(i)).$$

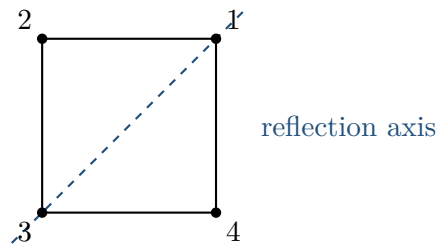
This holds for every i , proving that ρ is a homomorphism. □

Exercise 1.1. What is $\ker(\rho)$? Show that

$$\ker(\rho) = \{g \in G \mid g \circ s = s \text{ for all } s \in S\}. \quad (1.1)$$

2. DIHEDRAL GROUP D_8

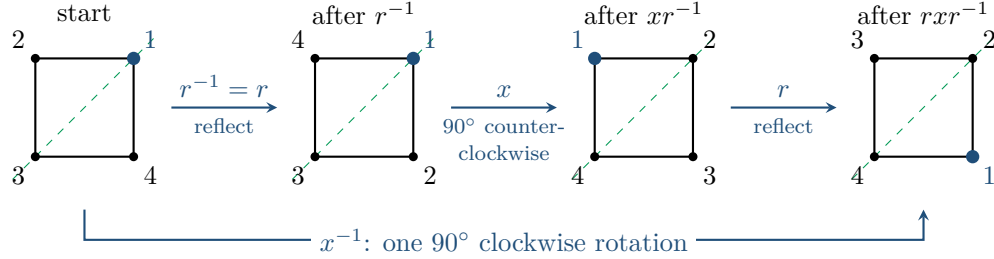
Label the vertices of a square consecutively as follows:



Counterclockwise rotation through 90° is the permutation $x = (1\ 2\ 3\ 4)$, and reflection across the diagonal through vertices 1 and 3 is $r = (2\ 4)$. They satisfy

$$x^4 = 1, \quad r^2 = 1, \quad r x r^{-1} = x^{-1}. \quad (2.1)$$

The last relation expresses the fact that reflecting a rotation reverses its direction. The picture below follows the vertex labels as we apply r^{-1} , then x , then r (products act from right to left). Since $r^{-1} = r$, both reflections use the dashed diagonal.



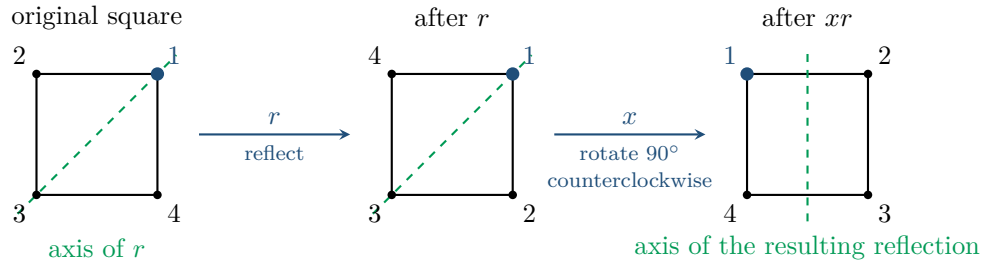
The final positions of all four labels agree with a clockwise quarter-turn of the original square, so $r x r^{-1} = x^{-1}$.

If the group D is generated by x, r and satisfies the relations (2.1), then in fact D consist of the following eight elements:

$$\langle x, r \rangle = \{1, x, x^2, x^3, r, x r, x^2 r, x^3 r\}.$$

Now we know that $1, x, x^2, x^3$ are rotations of the square, and r is a reflection. Do $x r, x^2 r, x^3 r$ also have geometric meanings? Yes! Let's compute these products from right to left and identify their reflection axes.

First example: understanding $x r$. Remember that $x r$ means *first reflect by $r = (2\ 4)$, then rotate counterclockwise by 90° using $x = (1\ 2\ 3\ 4)$* . The labels in the pictures move with the vertices. The blue vertex tracks the original vertex 1.



To compute the product, follow each original vertex through both steps:

$$\begin{aligned} 1 &\xrightarrow{r} 1 \xrightarrow{x} 2, & 2 &\xrightarrow{r} 4 \xrightarrow{x} 1, \\ 3 &\xrightarrow{r} 3 \xrightarrow{x} 4, & 4 &\xrightarrow{r} 2 \xrightarrow{x} 3. \end{aligned}$$

Thus $x r = (1\ 2)(3\ 4)$. Comparing the first and last squares, the left and right vertices have exchanged places while their heights are unchanged. Therefore $x r$ is reflection across the *vertical line through the midpoints of the top and bottom edges*.

The other two products: x^2r and x^3r . In each case we still apply r first. We then rotate by 180° for x^2r , or by 270° counterclockwise for x^3r . Following the vertices gives

product	$1 \mapsto$	$2 \mapsto$	$3 \mapsto$	$4 \mapsto$	cycle notation
x^2r	3	2	1	4	$(1\ 3)$
x^3r	4	3	2	1	$(1\ 4)(2\ 3)$

The pictures show the resulting reflections on the original labeled square. Dashed lines are the reflection axes; double arrows indicate exchanged vertices, and circles mark fixed vertices.



For x^2r , vertices 2 and 4 are fixed, so the reflection axis is exactly the diagonal joining them; vertices 1 and 3 are exchanged. For x^3r , the top and bottom vertices are exchanged while left and right positions are preserved, giving the horizontal reflection axis.

So we see that in D we have two types of elements: rotations and reflections.

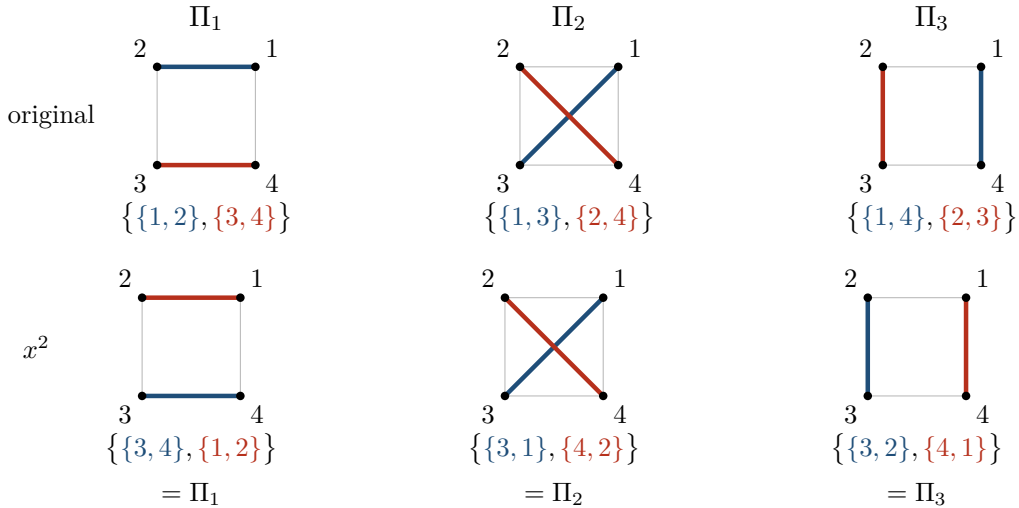
Note that D acts on $\{1, 2, 3, 4\}$ the vertices of the square and this gives us a homomorphism $\rho : D \rightarrow S_4$. What is its kernel? It must be trivial as only the identity element of D fixes all the vertices. So we can really think of D as a subset of S_4 if we like. If we do this,

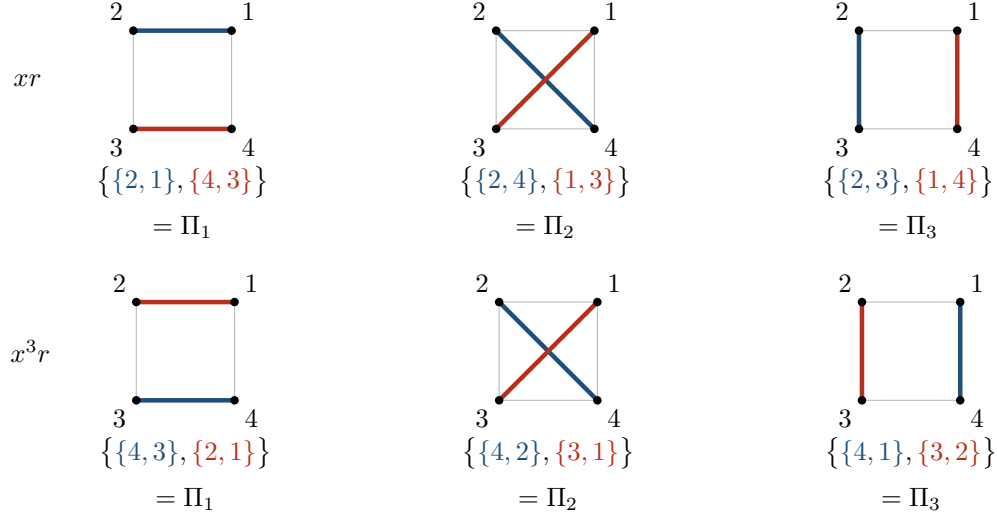
$$x^2 = (1\ 3)(2\ 4), \quad xr = (1\ 2)(3\ 4), \quad x^3r = (1\ 4)(2\ 3),$$

so this group contains the Klein four group K that came up as the kernel of $\varphi : S_4 \rightarrow S_3$.

You can also see this in pictures:

The three partitions are pictured in the top row. The other rows show what happens when we apply x^2 , xr , or x^3r to *every entry* of each partition. Vertex positions and labels are fixed in these pictures; the colored segments show the resulting pairs. Blue and red distinguish the two pairs so that we can track them.





The order within each pair, and the order of the two pairs, do not matter. For example, x^2 sends Π_1 to $\{\{3, 4\}, \{1, 2\}\} = \Pi_1$: the two pairs have simply exchanged places. Every column therefore remains the same partition. Thus

$$x^2 \circ \Pi_i = xr \circ \Pi_i = x^3r \circ \Pi_i = \Pi_i \quad (i = 1, 2, 3),$$

so all three symmetries lie in $\ker(\varphi) = K$.