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1. REVIEW: THE CORRESPONDENCE THEOREM

Recall the correspondence theorem from Lecture 15.

Let $\varphi : G \rightarrow \mathcal{G}$ be a *surjective* homomorphism, and set $K = \ker(\varphi) \trianglelefteq G$. For any subset $\mathcal{H} \subseteq \mathcal{G}$, its *preimage* under φ is

$$\varphi^{-1}(\mathcal{H}) = \{g \in G : \varphi(g) \in \mathcal{H}\}.$$

This notation does not require φ to have an inverse function.

Theorem 1.1 (Correspondence theorem).

(1) *There is a bijective correspondence*

$$\{\text{subgroups } H \leq G \text{ which contain } K\} \longleftrightarrow \{\text{subgroups } \mathcal{H} \leq \mathcal{G}\},$$

given by the mutually inverse maps

$$H \mapsto \varphi(H), \quad \mathcal{H} \mapsto \varphi^{-1}(\mathcal{H}).$$

(2) *Under this correspondence, $H \trianglelefteq G$ if and only if $\varphi(H) \trianglelefteq \mathcal{G}$.*

(3) *If H is finite, then $|H| = |K| |\varphi(H)|$.*

1.1. Proof of normality and orders. Suppose $H \trianglelefteq G$, and put $\mathcal{H} = \varphi(H)$. Given $x \in \mathcal{G}$, surjectivity gives $x = \varphi(g)$ for some $g \in G$. Hence

$$x\mathcal{H}x^{-1} = \varphi(g)\varphi(H)\varphi(g)^{-1} = \varphi(gHg^{-1}) = \varphi(H) = \mathcal{H}.$$

Therefore $\mathcal{H} \trianglelefteq \mathcal{G}$.

Conversely, suppose $\mathcal{H} \trianglelefteq \mathcal{G}$ and $H = \varphi^{-1}(\mathcal{H})$. For $g \in G$ and $h \in H$,

$$\varphi(ghg^{-1}) = \varphi(g)\varphi(h)\varphi(g)^{-1} \in \mathcal{H},$$

because $\varphi(h) \in \mathcal{H}$ and $\mathcal{H}$ is normal. Thus $ghg^{-1} \in H$ for every $g \in G$ and $h \in H$, proving $H \trianglelefteq G$. This proves part (2).

Finally, the restriction $\varphi|_H : H \rightarrow \mathcal{H}$ is surjective, and its kernel is $H \cap K = K$. The counting formula from Lecture 13 therefore gives

$$|H| = |\ker(\varphi|_H)| |\operatorname{im}(\varphi|_H)| = |K| |\mathcal{H}|,$$

when H is finite. This proves part (3).

2. REVIEW: CONSTRUCTION OF $\varphi : S_4 \rightarrow S_3$

Recall the construction from Lecture 15. There are exactly three ways to partition $\{1, 2, 3, 4\}$ into two unordered pairs:

$$\Pi_1 = \{\{1, 2\}, \{3, 4\}\},$$

$$\Pi_2 = \{\{1, 3\}, \{2, 4\}\},$$

$$\Pi_3 = \{\{1, 4\}, \{2, 3\}\}.$$

Every $\sigma \in S_4$ permutes these partitions by applying σ to each entry. Identifying Π_i with i defines

$$\varphi : S_4 \longrightarrow S_3, \quad \varphi(\sigma)(i) = j \quad \text{if } \sigma(\Pi_i) = \Pi_j.$$

Applying τ and then σ to the entries has the same effect as applying $\sigma\tau$. Hence $\varphi(\sigma\tau) = \varphi(\sigma)\varphi(\tau)$, so φ is a homomorphism.

Example 2.1. The permutation $(1\ 2\ 3\ 4)$ interchanges Π_1 and Π_3 and fixes Π_2 , while $(1\ 2)$ interchanges Π_2 and Π_3 and fixes Π_1 . Thus

$$\varphi((1\ 2\ 3\ 4)) = (1\ 3), \quad \varphi((1\ 2)) = (2\ 3).$$

These two transpositions generate S_3 , so φ is surjective. Also, $(1\ 2\ 3)$ sends $\Pi_1 \mapsto \Pi_3 \mapsto \Pi_2 \mapsto \Pi_1$, giving $\varphi((1\ 2\ 3)) = (1\ 3\ 2)$.

Each of $1, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)$ fixes all three partitions. The counting formula gives $|\ker(\varphi)| = 24/6 = 4$, so the kernel is precisely the Klein four group

$$K = \{1, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}.$$

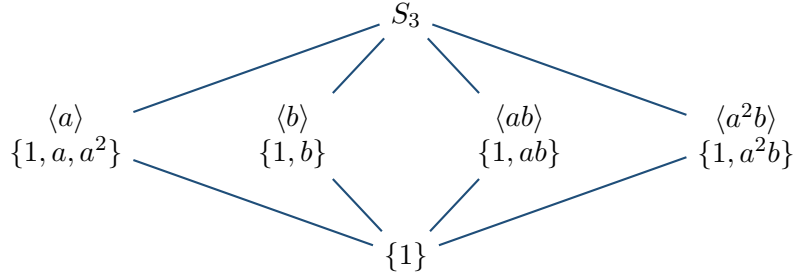
We can now use the correspondence theorem to study the subgroups of S_4 containing K .

3. APPLICATION TO $\varphi : S_4 \rightarrow S_3$

Write $S_3 = \{1, a, a^2, b, ab, a^2b\}$, where $a^3 = b^2 = 1$ and $bab^{-1} = a^{-1}$. Its subgroups are

$$\{1\}, \quad \langle a \rangle, \quad \langle b \rangle, \quad \langle ab \rangle, \quad \langle a^2b \rangle, \quad S_3.$$

Concretely, take $a = (1\ 2\ 3)$ and $b = (1\ 3)$. Then $a^2 = (1\ 3\ 2)$, $ab = (2\ 3)$, and $a^2b = (1\ 2)$, with permutation products composed from right to left. The subgroup inclusion graph is shown below; inclusions run upward.



Consequently, there are exactly six subgroups of S_4 containing K . Their orders follow from $|\varphi^{-1}(\mathcal{H})| = 4|\mathcal{H}|$:

$\mathcal{H} \leq S_3$	$\varphi^{-1}(\mathcal{H}) \leq S_4$	$ \varphi^{-1}(\mathcal{H}) $
$\{1\}$	K	4
$\langle a \rangle = \langle (1\ 2\ 3) \rangle$	A	12
$\langle b \rangle = \langle (1\ 3) \rangle$	H_1	8
$\langle ab \rangle = \langle (2\ 3) \rangle$	H_2	8
$\langle a^2b \rangle = \langle (1\ 2) \rangle$	H_3	8
S_3	S_4	24

What are the groups A, H_1, H_2, H_3 ?

First, we identify A with A_4 . Every element of K is an even permutation, so $K \leq A_4$. The restriction of φ to A_4 therefore has kernel K , and the counting formula gives

$$|\varphi(A_4)| = \frac{|A_4|}{|K|} = \frac{12}{4} = 3.$$

The unique subgroup of order 3 in S_3 is $\langle a \rangle$. Thus $\varphi(A_4) = \langle a \rangle$, and the correspondence theorem gives

$$A = \varphi^{-1}(\langle a \rangle) = A_4.$$

The three subgroups H_1, H_2, H_3 of order 8 will be identified with symmetry groups of squares in Lecture 17.