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1. SOME MOTIVATION

The next result we will study is quite abstract. It lets us learn about the structure of a group G using a homomorphism from G to another group. We begin with an example of a surjective homomorphism from S_4 to S_3 .

There are exactly three ways to partition $\{1, 2, 3, 4\}$ into two unordered pairs:

$$\Pi_1 = \{\{1, 2\}, \{3, 4\}\},$$

$$\Pi_2 = \{\{1, 3\}, \{2, 4\}\},$$

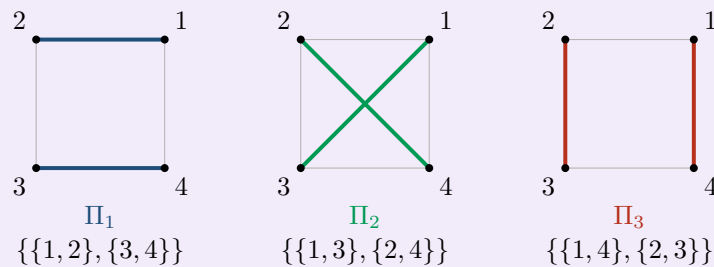
$$\Pi_3 = \{\{1, 4\}, \{2, 3\}\}.$$

Every $\sigma \in S_4$ permutes these three partitions by applying σ to each entry. Identifying Π_i with i gives a map

$$\varphi : S_4 \longrightarrow S_3.$$

This is a homomorphism: applying τ and then σ to the partitions has the same effect as applying $\sigma\tau$.

Remark (A picture of φ). Place 1, 2, 3, 4 at the vertices of a square. The three partitions are the three ways to join the vertices in two disjoint pairs. The colored segments below show those pairs:



In Π_2 , the pairs are opposite vertices; Π_1 and Π_3 use opposite edges. Any $\sigma \in S_4$ relabels the vertices: if $\sigma(\Pi_i) = \Pi_j$, then $\varphi(\sigma)(i) = j$. For the 90° counterclockwise rotation $\sigma = (1\,2\,3\,4)$,

$$\Pi_1 \mapsto \Pi_3, \quad \Pi_2 \mapsto \Pi_2, \quad \Pi_3 \mapsto \Pi_1.$$

so $\varphi(\sigma) = (1\,3)$.

Example 1.1. For $\sigma = (1\ 2\ 3\ 4)$, we obtain

$$\Pi_1 \mapsto \Pi_3, \quad \Pi_2 \mapsto \Pi_2, \quad \Pi_3 \mapsto \Pi_1.$$

Thus $\varphi((1\ 2\ 3\ 4)) = (1\ 3)$. Similarly, $\varphi((1\ 2)) = (2\ 3)$. Since $(1\ 3)$ and $(2\ 3)$ generate S_3 , the map φ is surjective.

Example 1.2. For $\sigma = (1\ 2\ 3)$, the vertex 4 is fixed. Applying σ to each entry of the partitions gives

$$\Pi_1 = \{\{1, 2\}, \{3, 4\}\} \mapsto \{\{2, 3\}, \{1, 4\}\} = \Pi_3,$$

$$\Pi_2 = \{\{1, 3\}, \{2, 4\}\} \mapsto \{\{2, 1\}, \{3, 4\}\} = \Pi_1,$$

$$\Pi_3 = \{\{1, 4\}, \{2, 3\}\} \mapsto \{\{2, 4\}, \{3, 1\}\} = \Pi_2.$$

Thus the induced permutation cycles the partitions as $\Pi_1 \mapsto \Pi_3 \mapsto \Pi_2 \mapsto \Pi_1$, and hence

$$\varphi((1\ 2\ 3)) = (1\ 3\ 2).$$

Recall the *counting formula for homomorphisms* from Lecture 13: if $\psi : A \rightarrow B$ is a homomorphism and A is finite, then

$$|A| = |\ker(\psi)| |\operatorname{im}(\psi)|.$$

In particular, if ψ is surjective, then $|A| = |\ker(\psi)| |B|$. We now apply this formula to the surjective homomorphism $\varphi : S_4 \rightarrow S_3$.

Example 1.3 (The kernel). Each of the permutations

$$1, \quad (1\ 2)(3\ 4), \quad (1\ 3)(2\ 4), \quad (1\ 4)(2\ 3)$$

fixes all three partitions.

The counting formula gives $|\ker(\varphi)| = 24/6 = 4$, so these are precisely the elements of the kernel:

$$K = \ker(\varphi) = \{1, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}.$$

This is the *Klein four group* inside S_4 . Each of its three nonidentity elements is a product of two disjoint transpositions, so its square is the identity and its order is 2. The identity has order 1. A cyclic group of order 4 must have a generator of order 4, but K has no such element. Therefore K is not cyclic.

2. THE CORRESPONDENCE THEOREM

Let $\varphi : G \rightarrow \mathcal{G}$ be a *surjective* homomorphism, and set $K = \ker(\varphi) \trianglelefteq G$. For any subset $\mathcal{H} \subseteq \mathcal{G}$, its *preimage* under φ is

$$\varphi^{-1}(\mathcal{H}) = \{g \in G : \varphi(g) \in \mathcal{H}\}.$$

This notation does not require φ to have an inverse function.

Theorem 2.1 (Correspondence theorem).

(1) *There is a bijective correspondence*

$$\{\text{subgroups } H \leq G \text{ which contain } K\} \longleftrightarrow \{\text{subgroups } \mathcal{H} \leq \mathcal{G}\},$$

given by the mutually inverse maps

$$H \mapsto \varphi(H), \quad \mathcal{H} \mapsto \varphi^{-1}(\mathcal{H}).$$

(2) *Under this correspondence, $H \trianglelefteq G$ if and only if $\varphi(H) \trianglelefteq \mathcal{G}$.*

(3) *If H is finite, then $|H| = |K| |\varphi(H)|$.*

2.1. The maps are well defined. For $H \leq G$ containing K , the image $\varphi(H)$ is a subgroup of $\mathcal{G}$: it is the image of the homomorphism $\varphi|_H : H \rightarrow \mathcal{G}$.

Conversely, let $\mathcal{H} \leq \mathcal{G}$. Then $\varphi^{-1}(\mathcal{H})$ is a subgroup of G . Indeed, $\varphi(e_G) = e_{\mathcal{G}} \in \mathcal{H}$, and if $a, b \in \varphi^{-1}(\mathcal{H})$, then

$$\varphi(ab) = \varphi(a)\varphi(b) \in \mathcal{H}.$$

Also $K \leq \varphi^{-1}(\mathcal{H})$, since every element of K maps to $e_{\mathcal{G}}$.

Exercise 2.2. Complete the subgroup verification by showing that $a \in \varphi^{-1}(\mathcal{H})$ implies $a^{-1} \in \varphi^{-1}(\mathcal{H})$.

2.2. The maps are inverse to each other. We claim that if $K \leq H \leq G$, then

$$\varphi^{-1}(\varphi(H)) = H. \tag{2.1}$$

The inclusion $H \subseteq \varphi^{-1}(\varphi(H))$ is immediate. For the reverse inclusion, suppose $x \in \varphi^{-1}(\varphi(H))$. Then $\varphi(x) = \varphi(a)$ for some $a \in H$, so

$$\varphi(xa^{-1}) = e_{\mathcal{G}}.$$

Thus $xa^{-1} \in K \leq H$, and $x = (xa^{-1})a \in H$.

Exercise 2.3. Use the surjectivity of φ to prove that $\varphi(\varphi^{-1}(\mathcal{H})) = \mathcal{H}$ for every $\mathcal{H} \leq \mathcal{G}$. Together with (2.1), this proves part (1).

2.3. Normality and orders. Suppose $H \trianglelefteq G$, and put $\mathcal{H} = \varphi(H)$. Given $x \in \mathcal{G}$, surjectivity gives $x = \varphi(g)$ for some $g \in G$. Hence

$$x\mathcal{H}x^{-1} = \varphi(g)\varphi(H)\varphi(g)^{-1} = \varphi(gHg^{-1}) = \varphi(H) = \mathcal{H}.$$

Therefore $\mathcal{H} \trianglelefteq \mathcal{G}$.

Exercise 2.4. Prove the converse: if $\mathcal{H} \trianglelefteq \mathcal{G}$, then $H = \varphi^{-1}(\mathcal{H}) \trianglelefteq G$.

Finally, the restriction $\varphi|_H : H \rightarrow \mathcal{H}$ is surjective, and its kernel is $H \cap K = K$. The counting formula from Lecture 13 therefore gives

$$|H| = |\ker(\varphi|_H)| |\operatorname{im}(\varphi|_H)| = |K| |\mathcal{H}|,$$

when H is finite. This proves part (3).