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1. LEFT AND RIGHT COSETS

Let $H \leq G$. Recall that the left and right cosets represented by $g \in G$ are

$$gH = \{gh : h \in H\}, \quad Hg = \{hg : h \in H\}.$$

So far we have mostly worked with left cosets. We now compare the two kinds of cosets.

In general, left cosets are *not* right cosets.

Example 1.1 (Left and right cosets in S_3). Write $S_3 = \{1, x, x^2, y, xy, x^2y\}$ with $x^3 = y^2 = 1$ and $yx = x^2y$, and let $H = \{1, y\}$. Then

$$\begin{aligned} H &= \{1, y\}, & H \cdot 1 &= \{1, y\}, \\ xH &= \{x, xy\}, & Hx &= \{x, yx\} = \{x, x^2y\}, \\ x^2H &= \{x^2, x^2y\}, & Hx^2 &= \{x^2, yx^2\} = \{x^2, xy\}. \end{aligned}$$

Here $yx^2 = (yx)x = x^2yx = x^2 \cdot x^2y = xy$. So $xH \neq Hx$ and $x^2H \neq Hx^2$; in fact neither xH nor x^2H is a right coset of H at all.

However, every statement you can make about left cosets has a right coset analogue, with the same proof after multiplying on the other side. For example:

- (1) Left cosets partition G ; so do right cosets.
- (2) $g_1H = g_2H$ if and only if $g_2^{-1}g_1 \in H$; similarly $Hg_1 = Hg_2$ if and only if $g_1g_2^{-1} \in H$.
- (3) $|gH| = |H|$, via $h \mapsto gh$; similarly $|Hg| = |H|$, via $h \mapsto hg$.

What about the index? Is the number of left cosets the same as the number of right cosets?

Proposition 1.1. *Let $H \leq G$. The map*

$$\{\text{left cosets of } H\} \longrightarrow \{\text{right cosets of } H\}, \quad gH \longmapsto Hg^{-1},$$

is a well-defined bijection.

Proof. For a subset $S \subseteq G$ write $S^{-1} = \{s^{-1} : s \in S\}$. Since $(gh)^{-1} = h^{-1}g^{-1}$ and $H^{-1} = H$,

$$(gH)^{-1} = H^{-1}g^{-1} = Hg^{-1}.$$

So the map is simply $S \mapsto S^{-1}$, which depends only on the set gH and not on the choice of g ; hence it is well defined. Applying the same operation to right cosets, $(Hg)^{-1} = g^{-1}H$, gives the inverse map. Thus it is a bijection. $\square$

Corollary 1.1. *The index $[G : H]$ equals both the number of left cosets and the number of right cosets of H in G . In particular, for finite G , $|G| = [G : H] |H|$ however we choose to count.*

Warning

The natural-looking map $gH \mapsto Hg$ is *not* well defined in general. In the example above, $xH = xyH$, but $Hx = \{x, x^2y\} \neq \{xy, x^2\} = Hxy$. (Check: $Hxy = \{xy, yxy\}$ and $yxy = x^2y^2 = x^2$.) This is why we use g^{-1} .

2. NORMAL SUBGROUPS AND COSETS

Recall that a subgroup $H \leq G$ is *normal*, written $H \trianglelefteq G$, if $gHg^{-1} \subseteq H$ for all $g \in G$, or equivalently $gHg^{-1} = H$ for all $g \in G$.

Although left cosets are not right cosets in general, they are for normal subgroups.

Lemma 2.1. *If $H \trianglelefteq G$, then every left coset of H is also a right coset. In fact, $gH = Hg$ for all $g \in G$.*

Proof. Multiplying $gHg^{-1} = H$ on the right by g gives $gH = Hg$. □

Lemma 2.2. *If a left coset gH equals some right coset Hg' , then in fact $Hg' = Hg$.*

Proof. Both the left cosets and the right cosets of H partition G . The left coset gH and the right coset Hg both contain $g = ge_G = e_Gg$, so $Hg' = gH$ meets Hg . Two right cosets that meet are equal, so $Hg' = Hg$. □

Corollary 2.1. *If every left coset of H is also some right coset, then H is normal in G .*

Proof. Let $g \in G$. By hypothesis gH is a right coset, so by Lemma 2.2, $gH = Hg$. Multiplying on the right by g^{-1} gives $gHg^{-1} = H$. This holds for all $g \in G$. □

Combining the last two results:

$$\begin{aligned} H \trianglelefteq G &\iff gH = Hg \text{ for all } g \in G \\ &\iff \text{every left coset of } H \text{ is a right coset.} \end{aligned}$$

For example, $H = \{1, y\}$ is not normal in S_3 , matching the computation $xH \neq Hx$ above.