

Lecture 10

MATH 327

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1. LEFT AND RIGHT COSETS

Definition 1.1 (Cosets and representatives). Let $H \leq G$ and $g \in G$. The *left coset* and *right coset* of H represented by g are

$$gH = \{gh : h \in H\}, \quad Hg = \{hg : h \in H\}. \quad (1.1)$$

We call g a *coset representative*. The set of all left cosets is

$$G/H = \{gH : g \in G\}. \quad (1.2)$$

A coset is a subset of G ; its representative is an element of G . Different representatives can give the same coset. If $g \in H$, then $gH = H$; we prove this in Lemma 2.1 below, and the same argument with multiplication on the right shows $Hg = H$.

Cosets give a new way to express normality. For a subgroup $N \leq G$ and $g \in G$,

$$gNg^{-1} = N \iff gN = Ng. \quad (1.3)$$

Why? Assume the right-hand side. Multiplying $gN = Ng$ on the right by g^{-1} gives $gNg^{-1} = Ng g^{-1} = N$. Conversely, if $gNg^{-1} = N$, then multiplying on the right by g gives $gN = Ng$. Thus N is normal exactly when every left coset of N equals the corresponding right coset.

In the examples below we write

$$S_3 = \{1, x, x^2, y, xy, x^2y\}, \quad x^3 = 1, \quad y^2 = 1, \quad yx = x^2y.$$

Example 1.1 (Cosets of the subgroup of order three in S_3). Take $N = \{1, x, x^2\}$. For its elements, we obtain

$$\begin{aligned} 1N &= \{1, x, x^2\} = N, \\ xN &= \{x, x^2, x^3\} = N, \\ x^2N &= \{x^2, x^3, x^4\} = N. \end{aligned}$$

For the other three elements, use $yx = x^2y$, $yx^2 = xy$, and $xyx = y$:

$$\begin{aligned} yN &= \{y, yx, yx^2\} = \{y, x^2y, xy\}, \\ xyN &= \{xy, xyx, xyx^2\} = \{xy, y, x^2y\} = yN, \\ x^2yN &= \{x^2y, x^2yx, x^2yx^2\} = \{x^2y, xy, y\} = yN. \end{aligned}$$

There are exactly two distinct left cosets:

$$S_3/N = \{N, yN\}, \quad S_3 = N \sqcup yN.$$

The symbol $\sqcup$ denotes a disjoint union. Since N is normal, the right coset represented by any g equals the corresponding left coset.

Example 1.2 (Cosets of a subgroup of order two in S_3). Now take the non-normal subgroup $H = \{1, y\}$. The left cosets are

$$\begin{aligned} H &= yH = \{1, y\}, \\ xH &= xyH = \{x, xy\}, \\ x^2H &= x^2yH = \{x^2, x^2y\}. \end{aligned}$$

Consequently,

$$S_3/H = \{H, xH, x^2H\}, \quad S_3 = H \sqcup xH \sqcup x^2H.$$

Here left and right cosets differ:

$$xH = \{x, xy\}, \quad Hx = \{x, yx\} = \{x, x^2y\}.$$

Since $xy \neq x^2y$, we have $xH \neq Hx$, consistent with the criterion above: H is not normal. Nevertheless, the left cosets still partition S_3 , as we now prove in general.

2. COSETS FORM A PARTITION

Definition 2.1 (Partition). A *partition* of a set X is a collection of nonempty subsets $\{X_\alpha\}_{\alpha \in I}$ such that

$$X = \bigcup_{\alpha \in I} X_\alpha, \quad X_\alpha \cap X_\beta = \emptyset \quad (\alpha \neq \beta).$$

We write $X = \bigsqcup_{\alpha \in I} X_\alpha$.

Lemma 2.1 (Equality of cosets). Let $H \leq G$ and $g_1, g_2 \in G$. The following are equivalent:

- (1) $g_1H \cap g_2H \neq \emptyset$;
- (2) $g_1H = g_2H$;
- (3) $g_2^{-1}g_1 \in H$;
- (4) $g_1^{-1}g_2 \in H$.

Proof. First consider the special case $g_2 = e_G$. We show that

$$gH \cap H \neq \emptyset \iff g \in H \iff gH = H.$$

We prove $g \in H \Rightarrow gH = H \Rightarrow gH \cap H \neq \emptyset \Rightarrow g \in H$.

- If $g \in H$, closure gives $gH \subseteq H$. Conversely, every $h \in H$ can be written $h = g(g^{-1}h)$, with $g^{-1}h \in H$, so $H \subseteq gH$. Thus $gH = H$.
- If $gH = H$, then $gH \cap H = H$, which is nonempty since $e_G \in H$.
- If $z \in gH \cap H$, write $z = gh$ with $h \in H$. Then $g = zh^{-1} \in H$.

This completes the special case.

For general g_1, g_2 , left multiplication by g_2^{-1} is a bijection of G . Hence

$$\begin{aligned} g_1H \cap g_2H \neq \emptyset &\iff (g_2^{-1}g_1)H \cap H \neq \emptyset \\ &\iff g_2^{-1}g_1 \in H \\ &\iff (g_2^{-1}g_1)H = H \\ &\iff g_1H = g_2H. \end{aligned}$$

Finally, $g_2^{-1}g_1 \in H$ if and only if its inverse $g_1^{-1}g_2$ belongs to H . □

Proposition 2.1. *The distinct left cosets of H form a partition of G .*

Proof. Every $g \in G$ belongs to gH , since $g = ge_G$ and $e_G \in H$. Thus the cosets are nonempty and cover G . By Lemma 2.1, two cosets that intersect are equal, so distinct cosets are disjoint. $\square$

Warning

G/H denotes the *set of left cosets*, not a set of representatives. If $R \subseteq G$ contains exactly one representative of each left coset, the precise formulas are

$$G/H = \{rH : r \in R\}, \quad G = \bigsqcup_{r \in R} rH. \quad (2.1)$$

The representative set R is not unique. For $N = \{1, x, x^2\}$ in S_3 , both $\{1, y\}$ and $\{x, xy\}$ are representative sets; for $H = \{1, y\}$, both $\{1, x, x^2\}$ and $\{y, xy, x^2y\}$ are. When H is normal, we will later give G/H a group operation.

3. EXAMPLE: COSETS OF $8\mathbb{Z}$

In the additive group $G = \mathbb{Z}$, take $H = 8\mathbb{Z}$. Cosets are written additively:

$$a + 8\mathbb{Z} = \{a + 8k : k \in \mathbb{Z}\}. \quad (3.1)$$

For instance,

$$0 + 8\mathbb{Z} = \{\dots, -16, -8, 0, 8, 16, \dots\},$$

$$1 + 8\mathbb{Z} = \{\dots, -15, -7, 1, 9, 17, \dots\},$$

$$7 + 8\mathbb{Z} = \{\dots, -9, -1, 7, 15, 23, \dots\}.$$

Arranging the integers by their remainders gives eight disjoint columns:

remainder	0	1	2	3	4	5	6	7
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	16	17	18	19	20	21	22	23
	8	9	10	11	12	13	14	15
	0	1	2	3	4	5	6	7
	-8	-7	-6	-5	-4	-3	-2	-1
	-16	-15	-14	-13	-12	-11	-10	-9
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Each column is one coset. The equality criterion becomes

$$a + 8\mathbb{Z} = b + 8\mathbb{Z} \iff a - b \in 8\mathbb{Z} \iff a \equiv b \pmod{8}. \quad (3.2)$$

For example, $0 + 8\mathbb{Z} = 8 + 8\mathbb{Z}$ and $1 + 8\mathbb{Z} = 17 + 8\mathbb{Z}$. Every integer has exactly one remainder in $\{0, 1, \dots, 7\}$, so

$$\mathbb{Z} = \bigsqcup_{r=0}^7 (r + 8\mathbb{Z}), \quad \mathbb{Z}/8\mathbb{Z} = \{0 + 8\mathbb{Z}, 1 + 8\mathbb{Z}, \dots, 7 + 8\mathbb{Z}\}.$$

Writing $\bar{a} = a + 8\mathbb{Z}$, we may abbreviate this as

$$\mathbb{Z}/8\mathbb{Z} = \{\bar{0}, \bar{1}, \dots, \bar{7}\}.$$

Thus an infinite group can have a finite set of cosets.