

CONTENTS

1. Homomorphisms, isomorphisms, and automorphisms.....	1
2. Example: conjugation	1
3. Normal subgroups.....	2

1. HOMOMORPHISMS, ISOMORPHISMS, AND AUTOMORPHISMS

We begin by recalling the definitions from Lecture 7. For groups G and H , a map $\varphi : G \rightarrow H$ is a homomorphism if

$$\varphi(ab) = \varphi(a)\varphi(b) \quad (a, b \in G).$$

It follows that $\varphi(e_G) = e_H$ and $\varphi(a^{-1}) = \varphi(a)^{-1}$.

Definition 1.1 (Isomorphism and automorphism). An *isomorphism* $\varphi : G \rightarrow H$ is a homomorphism admitting a homomorphic inverse $\psi : H \rightarrow G$, so that

$$\psi \circ \varphi = \text{id}_G, \quad \varphi \circ \psi = \text{id}_H.$$

We write $G \cong H$ if such a map exists, and $\text{Iso}(G, H)$ for the set of isomorphisms from G to H . An isomorphism $G \rightarrow G$ is called an *automorphism* of G ; the set of all automorphisms is denoted by $\text{Aut}(G)$.

As proved in Lecture 7 notes (*note:* we did this in Lecture 8 in class), for a homomorphism $\varphi : G \rightarrow H$,

$$\begin{aligned} \varphi \text{ is an isomorphism} &\iff \varphi \text{ is bijective} \\ &\iff \ker(\varphi) = \{e_G\} \text{ and } \text{im}(\varphi) = H. \end{aligned}$$

The kernel is a special kind of subgroup, called a normal subgroup. To explain this, we first introduce an important automorphism.

2. EXAMPLE: CONJUGATION

Definition 2.1 (Conjugation). Fix $g \in G$. The map

$$c_g : G \longrightarrow G, \quad c_g(a) = gag^{-1}, \tag{2.1}$$

is called *conjugation by g* .

If G is abelian, then $c_g(a) = a$ for every $a, g \in G$, so each c_g is the identity map. Conjugation becomes useful in studying nonabelian groups.

Proposition 2.1. For every $g \in G$, the map c_g is an automorphism, with inverse $c_{g^{-1}}$.

Proof. For $a, b \in G$, inserting $g^{-1}g = e_G$ gives

$$c_g(ab) = gabg^{-1} = ga(g^{-1}g)bg^{-1} = (gag^{-1})(gbg^{-1}) = c_g(a)c_g(b).$$

Thus c_g is a homomorphism. Moreover,

$$c_g(c_{g^{-1}}(a)) = g(g^{-1}ag)g^{-1} = a, \quad c_{g^{-1}}(c_g(a)) = g^{-1}(gag^{-1})g = a.$$

Hence $c_{g^{-1}}$ is its inverse. □

Remark. Conjugation distributes over any finite product:

$$g(a_1 \cdots a_r)g^{-1} = (ga_1g^{-1}) \cdots (ga_rg^{-1}).$$

Isomorphisms, including automorphisms, preserve properties expressed in terms of the group operation. For example, conjugate elements have the same order, since $(gag^{-1})^m = ga^mg^{-1}$ for every integer m .

Conjugation in cycle notation. Conjugation of permutations has a particularly simple description: it relabels the entries of each cycle.

Proposition 2.2. For $\sigma \in S_n$ and a cycle $\tau = (j_1 \cdots j_r)$,

$$\sigma\tau\sigma^{-1} = (\sigma(j_1) \cdots \sigma(j_r)). \quad (2.2)$$

Proof. Applying the three permutations from right to left sends

$$\sigma(j_i) \xrightarrow{\sigma^{-1}} j_i \xrightarrow{\tau} j_{i+1} \xrightarrow{\sigma} \sigma(j_{i+1}),$$

where $j_{r+1} = j_1$. Every entry outside $\{\sigma(j_1), \dots, \sigma(j_r)\}$ is fixed. This is exactly the cycle on the right-hand side. □

Since conjugation distributes over products, the same rule applies to every cycle in a disjoint-cycle decomposition. In particular, conjugation preserves cycle type.

3. NORMAL SUBGROUPS

Definition 3.1 (Normal subgroup). A subgroup $N \leq G$ is *normal* if

$$gNg^{-1} \subseteq N \quad \text{for every } g \in G, \quad (3.1)$$

where $gNg^{-1} = \{gng^{-1} : n \in N\}$. We write $N \trianglelefteq G$.

Remark. The inclusion is automatically an equality. Applying it to g^{-1} gives $g^{-1}Ng \subseteq N$. Conjugating this inclusion by g gives $N \subseteq gNg^{-1}$. Therefore $gNg^{-1} = N$ for every $g \in G$.

Example 3.1. Every subgroup of an abelian group is normal, since $gag^{-1} = a$. For any group G , the subgroups $\{e_G\}$ and G are normal. A normal subgroup different from both is called a *proper nontrivial normal subgroup*.

Definition 3.2 (Simple group). A nontrivial group G is *simple* if its only normal subgroups are $\{e_G\}$ and G .

Two cyclic subgroups of S_3 . Write

$$S_3 = \{1, x, x^2, y, xy, x^2y\}, \quad x^3 = 1, \quad y^2 = 1, \quad yx = x^2y.$$

Consider the cyclic subgroups

$$H = \langle x \rangle = \{1, x, x^2\}, \quad L = \langle y \rangle = \{1, y\}.$$

Both are abelian as groups in their own right. Are they normal in S_3 ?

Example 3.2 (The subgroup H is normal). Conjugation by any element of H fixes every element of H . To conjugate by y , use $y^{-1} = y$ and the relation $yx = x^2y$:

$$\begin{aligned} yxy^{-1} &= yxy = x^2, \\ yx^2y^{-1} &= (yxy^{-1})(yxy^{-1}) = x^2x^2 = x. \end{aligned}$$

Consequently,

$$\begin{aligned} yHy^{-1} &= \{y, yx, yx^2\}y^{-1} \\ &= \{1, yxy^{-1}, yx^2y^{-1}\} = \{1, x^2, x\} = H. \end{aligned}$$

This also deals with the remaining elements x^iy , because

$$(x^iy)H(x^iy)^{-1} = x^i(yHy^{-1})x^{-i} = x^iHx^{-i} = H.$$

Thus $H \trianglelefteq S_3$.

Example 3.3 (The subgroup L is not normal). It is enough to find one conjugation that does not preserve L . Since $x^{-1} = x^2$ and $xyx = y$, we have

$$xyx^{-1} = xyx^2 = (xyx)x = yx = x^2y.$$

Therefore

$$xLx^{-1} = \{1, xyx^{-1}\} = \{1, x^2y\} \neq \{1, y\} = L.$$

In particular, a subgroup can be abelian without being normal in its ambient group.

With $x = (1\ 2\ 3)$ and $y = (1\ 2)$, the six elements are

g	1	x	x^2	y	xy	x^2y
cycle notation	(1)	(1 2 3)	(1 3 2)	(1 2)	(1 3)	(2 3).

The subgroup H consists of the even permutations. We next construct the sign homomorphism and identify H as its kernel, giving another proof that H is normal.