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1. REVIEW OF LAST LECTURE

In the last lecture, we continued to study the permutation group S_n . We introduced the notion of a simple transposition

$$s_i := (i \ i+1) \text{ for } i = 1, \dots, n-1 \quad (1.1)$$

and then argued

Theorem 1.1. (1) Every $p \in S_n$ can be written as a product of simple transpositions

$$p = s_{i_1} \cdots s_{i_k}. \quad (1.2)$$

(2) The smallest number of simple transpositions needed to write p as in (1.2) is called the length of p , denoted $\ell(p)$, and is equal to the number of inversions in p . That is,

$$\ell(p) = \#\text{inv}(p), \quad \text{inv}(p) := \{(i, j) \mid 1 \leq i < j \leq n, p(i) > p(j)\}. \quad (1.3)$$

The key to proving property (2) was the following lemma, which I slightly misstated last time. The statement has been corrected in the notes, and a proof is now included.

Lemma 1.1. If $p \in S_n$ and $s_i = (i \ i+1)$ is a simple transposition, then

$$\#\text{inv}(ps_i) = \begin{cases} \#\text{inv}(p) + 1, & \text{if } p(i) < p(i+1), \\ \#\text{inv}(p) - 1, & \text{if } p(i) > p(i+1). \end{cases} \quad (1.4)$$

In particular, $\#\text{inv}(ps_i)$ differs from $\#\text{inv}(p)$ by exactly one.

Thus multiplying by a simple transposition changes the number of inversions by exactly one (an odd number).

Definition 1.1. A permutation $p \in S_n$ to be *even* if it can be written as a product of an even number of simple transpositions.

Warning

Why is this a well-defined concept ? In other words, can a permutation be written both as an even and an odd number of simple transpositions ?

No! Start with an easy case. Can we write the trivial permutation using an odd number of simple transpositions, i.e. $(1) = s_{i_1} \cdots s_{i_{2m+1}}$? By the Lemma we can think of the right hand side as

starting from $p = (1)$ and then applying a sequence of $2m + 1$ simple transpositions, each of which changes the number of inversions by ± 1 . So the number of inversions would have to be odd (imagine starting at 0 and adding/subtracting 1 7 times...). On the other hand, this number of inversions must also be equal to the number of inversions of (0) so it has to be 0, a contradiction.

Now for the general case: say $p = s_{i_1} \dots s_{i_k}$ with k even and also $p = s_{j_1} \dots s_{j_d}$ with d odd. Then $p^{-1} = s_{j_d} \dots s_{j_1}$ (Why?) So then

$$(1) = pp^{-1} = s_{i_1} \dots s_{i_k} s_{j_d} \dots s_{j_1}. \quad (1.5)$$

So by what we just said $d + k$ is even, which means that d and k have the same parity.

Denote the set of even permutations by

$$A_n := \{p \in S_n \mid p \text{ is even}\}. \quad (1.6)$$

Equivalently, $p \in A_n$ if and only if $\ell(p)$ is even.

Proposition 1.1. A_n is a subgroup of S_n

Proof. Since S_n is finite, it suffices to check that A_n contains the identity and is closed under multiplication. The identity is a product of zero simple transpositions, so it lies in A_n . If $p, q \in A_n$, choose expressions for p and q involving even numbers of simple transpositions. Concatenating these expressions gives an expression for pq involving an even number of simple transpositions. Hence $pq \in A_n$. $\square$

We will come back and give a different proof of this result a bit later.

Exercise 1.2. For $n \geq 2$, how many even permutations are there?

2. THE GROUP S_3

The six elements of S_3 are the identity, the three transpositions, and the two 3-cycles:

$$S_3 = \{\text{id}, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}. \quad (2.1)$$

Just to connect with our previous example, let us note that the lengths of these elements are as follows:

p	id	(1 2)	(2 3)	(1 2 3)	(1 3 2)	(1 3)
$\ell(p)$	0	1	1	2	2	3

(2.2)

Indeed, if $s_1 = (1\ 2)$ and $s_2 = (2\ 3)$, then

$$(1\ 2\ 3) = s_1 s_2, \quad (1\ 3\ 2) = s_2 s_1, \quad (1\ 3) = s_1 s_2 s_1. \quad (2.3)$$

The two 3-cycles are neither the identity nor simple transpositions, so their length is at least 2, and the displayed expressions show it is exactly 2. Finally, $(1\ 3)$ is not a simple transposition, and it cannot have length 2: the only products of two simple transpositions in S_3 are $s_1 s_1 = s_2 s_2 = \text{id}$, $s_1 s_2 = (1\ 2\ 3)$ and $s_2 s_1 = (1\ 3\ 2)$. Thus its length is 3.

Set

$$x = (1\ 2\ 3) \quad \text{and} \quad y = (1\ 3). \quad (2.4)$$

Then

$$x^2 = (1\ 3\ 2), \quad x^3 = \text{id}, \quad y^2 = \text{id}, \quad (2.5)$$

and direct calculation gives

$$yx = x^2y. \quad (2.6)$$

Consequently,

$$S_3 = \{\text{id}, x, x^2, y, xy, yx\}. \quad (2.7)$$

The table below gives the multiplication in S_3 . Its entry in row g and column h is gh , so h is applied first.

$\cdot$	id	x	x^2	y	xy	yx
id	id	x	x^2	y	xy	yx
x	x	x^2	id	xy	yx	y
x^2	x^2	id	x	yx	y	xy
y	y	yx	xy	id	x^2	x
xy	xy	y	yx	x	id	x^2
yx	yx	xy	y	x^2	x	id

(2.8)

Cayley graph of S_3 .

A Cayley graph gives a geometric picture of a group together with a chosen set of ‘generators.’ We use

$$x = (1\ 2\ 3) \quad \text{and} \quad y = (1\ 3). \quad (2.9)$$

The vertices are the six elements of S_3 . From each vertex g , a blue arrow leads to gx , and a red edge joins g to gy . The red edges need no arrows because $y^2 = \text{id}$: traversing a red edge twice returns to the starting vertex. The blue edges form two directed triangles, and the red edges join corresponding vertices of the two triangles.

