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1. TRANSPOSITIONS AND LENGTH

In the last lecture, we saw that every permutation $p \in S_n$ can be decomposed into disjoint cycles. We also introduced the following notion:

Definition 1.1. A simple transposition in S_n is a permutation of the form $s_i := (i \ i + 1)$ for $i = 1, 2, \dots, n - 1$.

For example, in S_3 the transpositions $s_1 = (1 \ 2)$ and $s_2 = (2 \ 3)$ are simple, but $(1 \ 3)$ is not.

Using a variant of the bubble sort idea, one can now prove

Theorem 1.1. Every $p \in S_n$ can be written as a product of simple transpositions s_i for $i = 1, \dots, n - 1$. The minimal number of such transpositions needed to write p is called the (Coxeter) length of p and is denoted as $\ell(p)$.

Warning

The length of a *simple* transposition $(i \ i + 1)$ is always one, but this isn't the case for an arbitrary transposition $(a \ b)$. For example, take

$$\tau = (1 \ 3) \in S_3. \quad (1.1)$$

In one-line notation, $\tau = [3, 2, 1]$, and bubble sort gives

$$[3, 2, 1] \xrightarrow{s_1} [2, 3, 1] \xrightarrow{s_2} [2, 1, 3] \xrightarrow{s_1} [1, 2, 3]. \quad (1.2)$$

Thus τ has length 3, not 1. Equivalently,

$$\tau = s_1 s_2 s_1. \quad (1.3)$$

2. INVERSIONS AND LENGTH

Definition 2.1 (Inversion). Let $p \in S_n$. An *inversion* of p is a pair (i, j) such that $1 \leq i < j \leq n$ and $p(i) > p(j)$. The *inversion number* of p is the number of such pairs:

$$\text{inv}(p) := \#\{(i, j) : 1 \leq i < j \leq n, p(i) > p(j)\}. \quad (2.1)$$

Example 2.1. Consider $p \in S_4$ the permutation corresponding to the unordered list $[3, 1, 4, 2]$. Its inversions correspond to the pairs

$$(1, 2), \quad (1, 4), \quad (3, 4). \quad (2.2)$$

Therefore

$$\text{inv}(p) = 3. \quad (2.3)$$

For comparison, $(2, 4)$ is not an inversion because $p(2) = 1 < p(4) = 2$.

Lemma 2.1. *If $p \in S_n$ and $s_i = (i \ i + 1)$ is a simple transposition, then*

$$\text{inv}(ps_i) = \begin{cases} \text{inv}(p) + 1, & \text{if } p(i) < p(i + 1), \\ \text{inv}(p) - 1, & \text{if } p(i) > p(i + 1). \end{cases} \quad (2.4)$$

In particular, $\text{inv}(ps_i)$ differs from $\text{inv}(p)$ by exactly one.

Proof. Let $I(p) := \{(i, j) : 1 \leq i < j \leq n, p(i) > p(j)\}$ be the set whose cardinality is the inversion number. Then we want to compare $I(p)$ with $I(ps_i)$, i.e., we want to compare the number of inversions of the list corresponding to p and that corresponding to ps_i (the list obtained from p by swapping the i -th and $(i + 1)$ -st entries). If $p(i) > p(i + 1)$, then ps_i corresponds to a ‘desirable’ move from the bubble sort perspective, in that it moves the list closer to being in order. Hence we expect the number of inversions to go down.

More precisely, if $1 \leq u < v < i$, then the operation $p \mapsto ps_i$ does nothing to the first $i - 1$ elements of the list, so (u, v) belongs to $I(ps_i)$ if and only if it belongs to $I(p)$. If $i + 1 < u < v \leq n$, again as $p \mapsto ps_i$ does nothing to the entries after $i + 1$ in the list, (u, v) has the same inversion status in $I(p)$ and $I(ps_i)$. The remaining pairs are handled as follows.

- (1) If $u < i$ and $v > i + 1$, then the operation $p \mapsto ps_i$ does not change $p(u)$ or $p(v)$, so again (u, v) belongs to $I(ps_i)$ if and only if it belongs to $I(p)$.
- (2) Consider $(i, i + 1)$. Now clearly depending on whether we are in the case $p(i) < p(i + 1)$ or $p(i) > p(i + 1)$, the pair will either appear or disappear respectively in $I(ps_i)$.
- (3) If $u = i$ and $v = i + 1 + h$ with $h > 0$, then $(i, i + 1 + h) \in I(p)$ if and only if, after the swap of the i -th and $(i + 1)$ -st entries, we have $(i + 1, i + 1 + h) \in I(ps_i)$. Similarly, $(i + 1, i + 1 + h) \in I(p)$ if and only if $(i, i + 1 + h) \in I(ps_i)$. Thus the total number of inversions contributed by these two pairs is unchanged.
- (4) If $u < i$, then similarly $(u, i) \in I(p)$ if and only if $(u, i + 1) \in I(ps_i)$, and $(u, i + 1) \in I(p)$ if and only if $(u, i) \in I(ps_i)$. Again the total number of inversions contributed by these two pairs is unchanged.

Thus the only change in the total number of inversions comes from the pair $(i, i + 1)$: the number increases by one if $p(i) < p(i + 1)$ and decreases by one if $p(i) > p(i + 1)$, as claimed. $\square$

Example 2.2. For the permutation $p = [3, 1, 4, 2]$ above, we have $\text{inv}(p) = 3$. Since $p(1) > p(2)$, right multiplication by s_1 decreases the inversion number:

$$ps_1 = [1, 3, 4, 2], \quad \text{inv}(ps_1) = 2. \quad (2.5)$$

On the other hand, $p(2) < p(3)$, so right multiplication by s_2 increases it:

$$ps_2 = [3, 4, 1, 2], \quad \text{inv}(ps_2) = 4. \quad (2.6)$$

Thus this example illustrates both cases of the lemma.

Theorem 2.1. *For every permutation $p \in S_n$, its Coxeter length is equal to its inversion number:*

$$\ell(p) = \text{inv}(p). \quad (2.7)$$

Proof. Choose a shortest expression $p = s_{i_1} \cdots s_{i_{\ell(p)}}$. Starting from id and multiplying by these simple transpositions one at a time, the lemma shows that the inversion number changes by exactly one at each step. Since $\text{inv}(\text{id}) = 0$, it follows that $\text{inv}(p) \leq \ell(p)$.

Conversely, use the bubble-sort factorization from Theorem 1.1. Each step swaps an adjacent pair $p(i) > p(i+1)$, so the lemma shows that it decreases the inversion number by exactly one. The process ends at the identity, whose inversion number is zero, and therefore uses exactly $\text{inv}(p)$ simple transpositions. Hence $\ell(p) \leq \text{inv}(p)$, proving the equality. $\square$