

Homework 3

MATH 327

Fall 2026

Covers Lectures 15–19

Suggested completion date: Monday, October 26

Unless otherwise indicated, the following exercises are from Michael Artin's *Algebra*, second edition.

Chapter 2: Groups

Section 10: The Correspondence Theorem

Exercise 10.5 (Adapted)

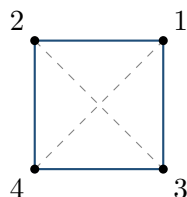
Let $\varphi: S_4 \rightarrow S_3$ be the homomorphism described in Example 2.5.13. In class we determined the subgroups

$$A = \varphi^{-1}(\langle(123)\rangle) \quad \text{and} \quad H_1 = \varphi^{-1}(\langle(13)\rangle).$$

Perform a similar analysis to determine the subgroup

$$H_2 = \varphi^{-1}(\langle(12)\rangle).$$

Hint. Relabel the vertices of the square as shown below and consider its geometric symmetries as permutations in S_4 .



Exercise S.1 (Supplementary)

Prove that there is no surjective homomorphism $S_5 \rightarrow S_4$.

Dihedral Groups and Group Actions

Exercise 1 (Supplementary)

Let $n \geq 3$. Define D_{2n} by the presentation

$$D_{2n} = \langle x, y \mid x^n = 1, y^2 = 1, yx = x^{n-1}y \rangle.$$

(a) Prove that every element of D_{2n} can be written uniquely in one of the forms

$$x^k \quad \text{or} \quad x^k y, \quad 0 \leq k < n.$$

Conclude that D_{2n} has exactly $2n$ elements.

(b) Let $H := \{1, x, \dots, x^{n-1}\}$. Show that H is a normal subgroup of D_{2n} .

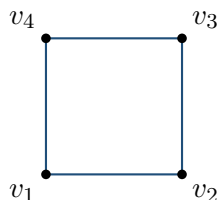
(c) Let $I := \{1, y\}$. Is I normal in D_{2n} ? Justify your answer.

Exercise 2 (Supplementary)

Let a group G act on a set S . Recall that the *orbit* of $s \in S$ and its *stabilizer* are defined by

$$O_s := \{g \circ s \mid g \in G\}, \quad \text{Stab}_G(s) := \{g \in G \mid g \circ s = s\}.$$

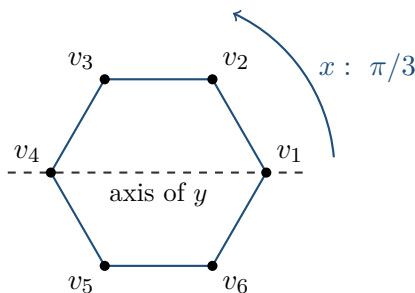
Consider the natural action of D_8 , the symmetry group of the square, on its vertices $V = \{v_1, v_2, v_3, v_4\}$, labeled as follows.



Determine the orbit O_{v_1} and the stabilizer $\text{Stab}_{D_8}(v_1)$.

Exercise 3 (Supplementary)

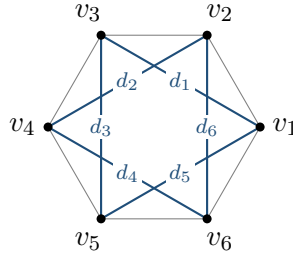
Consider the natural action of D_{12} on the vertices $V = \{v_1, \dots, v_6\}$ of a regular hexagon, labeled counterclockwise as shown below. Let x be counterclockwise rotation through $\pi/3$, and let y be reflection across the diagonal joining v_1 and v_4 .



(a) Describe geometrically each of the transformations xy , x^2y , x^3y , x^4y , and x^5y . Identify the axis of any reflection that occurs. Here products act from right to left, so xy means

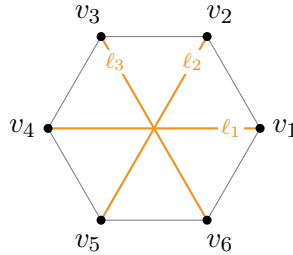
first apply y , then x .

- (b) Describe the homomorphism $\rho: D_{12} \rightarrow S_6$ induced by this action by writing $\rho(x)$ and $\rho(y)$ in cycle notation. Determine $\ker \rho$.
- (c) **Short diagonals.** Let $\mathcal{D}_s = \{d_1, \dots, d_6\}$, where $d_i = \overline{v_i v_{i+2}}$ and vertex subscripts are read modulo 6. A diagonal is an unordered pair of endpoints, so a symmetry may preserve a diagonal while exchanging its endpoints. The short diagonals are shown in blue below.



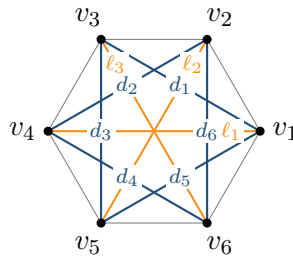
Determine the orbits of D_{12} on $\mathcal{D}_s$ and compute $\text{Stab}_{D_{12}}(d_1)$. Describe the induced homomorphism $\rho_s: D_{12} \rightarrow S_6$ by giving $\rho_s(x)$ and $\rho_s(y)$ in cycle notation, using $d_i \leftrightarrow i$. Determine $\ker \rho_s$.

- (d) **Long diagonals.** Let $\mathcal{D}_\ell = \{\ell_1, \ell_2, \ell_3\}$, where $\ell_i = \overline{v_i v_{i+3}}$. These diagonals join opposite vertices and are shown in orange below.



Determine the orbits of D_{12} on $\mathcal{D}_\ell$ and compute $\text{Stab}_{D_{12}}(\ell_1)$. Describe the induced homomorphism $\rho_\ell: D_{12} \rightarrow S_3$ by giving $\rho_\ell(x)$ and $\rho_\ell(y)$ in cycle notation, using $\ell_i \leftrightarrow i$. Determine $\ker \rho_\ell$.

- (e) **All diagonals.** Let $\mathcal{D} = \mathcal{D}_s \cup \mathcal{D}_\ell$ be the set of all nine diagonals, shown below.



Determine the orbits of D_{12} on $\mathcal{D}$ and compute the stabilizer of a representative from each orbit. Describe the induced homomorphism $\rho_{\mathcal{D}}: D_{12} \rightarrow S_9$ by giving the images of x and y in cycle notation, using

$$(d_1, d_2, d_3, d_4, d_5, d_6, \ell_1, \ell_2, \ell_3) \longleftrightarrow (1, 2, 3, 4, 5, 6, 7, 8, 9).$$

Determine $\ker \rho_{\mathcal{D}}$.

Section 11: Product Groups

Exercise 11.6

Let G be a group that contains normal subgroups of orders 3 and 5, respectively. Prove that G contains an element of order 15.

Exercise 11.9

Let H and K be subgroups of a group G . Prove that the product set HK is a subgroup of G if and only if $HK = KH$.

Section 12: Quotient Groups

Exercise 12.4

Let $H = \{\pm 1, \pm i\}$ be the subgroup of $G = \mathbb{C}^\times$ of fourth roots of unity. Describe the cosets of H in G explicitly. Is G/H isomorphic to G ?

Exercise 12.5

Let G be the group of upper triangular real matrices $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ with a and d different from zero.

For each of the following subsets, determine whether or not S is a subgroup, and whether or not S is a normal subgroup. If S is a normal subgroup, identify the quotient group G/S .

- (i) S is the subset defined by $b = 0$.
- (ii) S is the subset defined by $d = 1$.
- (iii) S is the subset defined by $a = d$.