

Homework 2

MATH 327

Fall 2026

Questions and solutions

Covers Lectures 8–13

Suggested completion date: October 5

Unless otherwise indicated, the following exercises are from Michael Artin's *Algebra*, second edition.

Chapter 2: Groups

Section 8: Cosets

Exercise 8.1

Let H be the cyclic subgroup of the alternating group A_4 generated by the permutation (123) . Exhibit the left and the right cosets of H explicitly.

Solution

The group A_4 has 12 elements and $H = \langle (123) \rangle$ has 3, so there are four left cosets and four right cosets. Direct computation gives the left cosets

$$\begin{aligned}H &= \{1, (123), (132)\}, \\(12)(34)H &= \{(12)(34), (143), (243)\}, \\(13)(24)H &= \{(13)(24), (142), (234)\}, \\(14)(23)H &= \{(14)(23), (124), (134)\},\end{aligned}$$

and the right cosets

$$\begin{aligned}H &= \{1, (123), (132)\}, \\H(12)(34) &= \{(12)(34), (134), (234)\}, \\H(13)(24) &= \{(13)(24), (124), (243)\}, \\H(14)(23) &= \{(14)(23), (142), (143)\}.\end{aligned}$$

Exercise 8.4

Does a group of order 35 contain an element of order 5? Of order 7?

Solution

If G has an element x of order 35, then x^7 has order 5 and x^5 has order 7. Otherwise, to show that G has an element of order 7, suppose every nonidentity element has order 5. Distinct subgroups of order 5 intersect only in the identity, so counting their elements would give $35 = 1 + 4k$ for some integer k , a contradiction. The same argument shows that G has an element of order 5: if every nonidentity element had order 7, then $35 = 1 + 6k$, also impossible.

Exercise 8.10

Prove that every subgroup of index 2 is a normal subgroup, and show by example that a subgroup of index 3 need not be normal.

Solution

Suppose $[G : H] = 2$, and let $a \notin H$. Since $aH \neq H$ and $Ha \neq H$, the two-coset decompositions of G give

$$G = H \sqcup aH = H \sqcup Ha.$$

Thus $aH = Ha$ for every $a \notin H$; the same equality holds for $a \in H$. Hence H is normal in G . For an index-3 example, take $G = S_3$ and $H = \{1, (12)\}$. Then $[G : H] = 3$, but $(123)(12)(132) = (23) \notin H$, so H is not normal.

Section 9: Modular Arithmetic

Exercise 9.6

Prove the Chinese Remainder Theorem: Let a, b, u, v be integers, and assume that the greatest common divisor of a and b is 1. Then there is an integer x such that

$$x \equiv u \pmod{a} \quad \text{and} \quad x \equiv v \pmod{b}.$$

Hint. Do the case $u = 0$ and $v = 1$ first.

Solution

Since $\gcd(a, b) = 1$, choose integers r, s with $ra + sb = 1$. Set $x = ra$ and $y = sb$. Then

$$x \equiv 0 \pmod{a}, \quad x \equiv 1 \pmod{b}, \quad y \equiv 1 \pmod{a}, \quad y \equiv 0 \pmod{b}.$$

Therefore $uy + vx$ satisfies both required congruences.

Miscellaneous Problems

Exercise M.12

Postage stamp problem. Let a and b be positive, relatively prime integers.

- (a) Prove that every sufficiently large positive integer n can be obtained as $ra + sb$, where r and s are positive integers.
- (b) Determine the largest integer that is not of this form.

Solution

Since $\gcd(a, b) = 1$, for any integer n there is a unique $r \in \{1, \dots, b\}$ such that $ra \equiv n \pmod{b}$. Thus

$$s = \frac{n - ra}{b}$$

is an integer. If $n > ab$, then $ra \leq ab < n$, so $s > 0$; also $r > 0$. Hence every integer $n > ab$ has the required form $ra + sb$.

The integer ab itself does not have this form with $r, s > 0$. Indeed, if $ab = ra + sb$, then reducing

modulo b gives $b \mid ra$. Since $\gcd(a, b) = 1$, we have $b \mid r$, so $r \geq b$ and $ra \geq ab$, contradicting $sb > 0$. Consequently, the largest integer not of the required form is $\boxed{ab}$.