

Unless otherwise indicated, the following exercises are from Michael Artin's *Algebra*, second edition.

Chapter 1: Matrices

Section 1: The Basic Operations

Exercise 1.13

A square matrix A is *nilpotent* if $A^k = 0$ for some $k > 0$. Prove that if A is nilpotent, then $I + A$ is invertible. Do this by finding the inverse.

Miscellaneous Problems

Exercise M.10

Let A, B be $m \times n$ and $n \times m$ matrices. Prove that $I_m - AB$ is invertible if and only if $I_n - BA$ is invertible.

Hint. Perhaps the only approach available to you at this time is to find an explicit expression for one inverse in terms of the other. As a heuristic tool, you could try substituting into the power series expansion for $(1 - x)^{-1}$. The substitution will make no sense unless some series converge, and this needn't be the case. But any way to guess a formula is permissible, provided that you check your guess afterward.

Chapter 2: Groups

Supplementary Problems: Permutations, Cycles, and Length

Exercise P.1

Let $p \in S_6$ be the permutation written in 'one-line notation' as

$$p = [4, 1, 6, 2, 5, 3].$$

- (a) Write p as a product of disjoint cycles.
- (b) List all inversions of p .
- (c) Compute $\text{inv}(p)$ and hence determine $\ell(p)$.
- (d) Express p as a product of exactly $\ell(p)$ simple transpositions.

Exercise P.2

Let $1 \leq a < b \leq n$, and consider the transposition

$$\tau = (a \ b) \in S_n.$$

(a) Write τ in one-line notation.

(b) Prove that

$$\text{inv}(\tau) = 2(b - a) - 1.$$

(c) Deduce that $\ell(\tau) = 2(b - a) - 1$.

(d) Use this result to explain why two permutations with the same cycle type need not have the same Coxeter length.

Section 3: Subgroups of the Additive Group of Integers

Exercise 3.3

(a) Define the greatest common divisor of a set $\{a_1, \dots, a_n\}$ of n integers. Prove that it exists, and that it is an integer combination of $a_1, \dots, a_n$.

(b) Prove that if the greatest common divisor of $\{a_1, \dots, a_n\}$ is d , then the greatest common divisor of $\{a_1/d, \dots, a_n/d\}$ is 1.

Section 4: Cyclic Groups

Exercise 4.3

Let a and b be elements of a group G . Prove that ab and ba have the same order.

Exercise 4.7

Let x and y be elements of a group G . Assume that each of the elements x , y , and xy has order 2. Prove that the set $H = \{1, x, y, xy\}$ is a subgroup of G , and that it has order 4.

Section 5: Homomorphisms

Exercise 5.3

Let U denote the group of invertible upper triangular 2×2 matrices $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, and let $\varphi: U \rightarrow \mathbb{R}^\times$ be the map that sends $A \mapsto a^2$. Prove that φ is a homomorphism, and determine its kernel and image.

Exercise 5.6

Determine the center of $\text{GL}_n(\mathbb{R})$.

Hint. You are asked to determine the invertible matrices A that commute with every invertible matrix B . Do not test with a general matrix B . Test with elementary matrices.

Section 6: Isomorphisms

Exercise 6.7

Let H be a subgroup of G , and let g be a fixed element of G . The *conjugate subgroup* gHg^{-1} is defined to be the set of all conjugates ghg^{-1} , with h in H . Prove that gHg^{-1} is a subgroup of G .

Exercise 6.11

Let a be an element of a group G . Prove that if the set $\{1, a\}$ is a normal subgroup of G , then a is in the center of G .