

Extra practice for Midterm

MATH 327: Algebra 1
Midterm 1

Fall 2026
Solutions

The first three exercises are from Michael Artin's *Algebra*, second edition, Chapter 2, Section 8 (Cosets).

Exercise 8.5

A finite group G contains an element x of order 10 and also an element y of order 6. What can be said about the order of G ?

Solution

By Lagrange's theorem, the orders of $\langle x \rangle$ and $\langle y \rangle$ divide $|G|$. Hence $10 \mid |G|$ and $6 \mid |G|$, so

$$30 \mid |G|.$$

In particular, $|G| \geq 30$. Every positive multiple of 30 can occur: if $G = \langle g \rangle$ is cyclic of order $30m$, then g^{3m} has order 10 and g^{5m} has order 6.

Exercise 8.7

A group G of order 22 contains elements x and y , where $x \neq 1$ and y is not a power of x . Prove that the subgroup generated by these elements is the whole group G .

Solution

Let $H = \langle x, y \rangle$. Since $H \leq G$, Lagrange's theorem gives $|H| \in \{1, 2, 11, 22\}$. Because $x \neq 1$, we have $|H| > 1$.

If $|H|$ were 2 or 11, then H would have prime order. The nonidentity element $x \in H$ would therefore generate H : its order divides $|H|$ and is greater than 1, so it equals $|H|$. But then $y \in H = \langle x \rangle$ would be a power of x , contrary to the hypothesis. Thus $|H| = 22$, and consequently $H = G$.

Exercise 8.8

Let G be a group of order 25. Prove that G has at least one subgroup of order 5, and that if it contains only one subgroup of order 5, then it is a cyclic group.

Solution

Choose $a \neq 1$ in G . By Lagrange's theorem, its order is either 5 or 25. If its order is 5, then $\langle a \rangle$ is a subgroup of order 5. If its order is 25, then a^5 has order 5: $(a^5)^5 = 1$, while $a^5 \neq 1$, so its order is 5. Thus G always has a subgroup of order 5.

Now suppose H is the unique subgroup of order 5. Since $|H| < |G|$, choose $b \in G \setminus H$. In particular, $b \neq 1$, so its order is 5 or 25. If its order were 5, then $\langle b \rangle$ would be a subgroup of order 5 and hence would equal H , contradicting $b \notin H$. Therefore b has order 25, so $G = \langle b \rangle$ is cyclic.

Short-answer practice

Give the requested answers; no proof is required. Compose permutations rightmost first.

Exercise S1

In the additive group $\mathbb{Z}/20\mathbb{Z}$, find the order of $\bar{8}$ and list the elements of $\langle \bar{8} \rangle$.

Solution

The order is 5, and

$$\langle \bar{8} \rangle = \{\bar{0}, \bar{4}, \bar{8}, \bar{12}, \bar{16}\}.$$

Indeed, $k\bar{8} = \bar{0}$ exactly when $20 \mid 8k$, or equivalently $5 \mid k$.

Exercise S2

Let $\varphi : \mathbb{Z} \rightarrow \mathbb{Z}/12\mathbb{Z}$ be the homomorphism $\varphi(n) = \overline{4n}$. Describe $\ker \varphi$ and list the elements of $\operatorname{im} \varphi$.

Solution

We have $\ker \varphi = 3\mathbb{Z}$, because $12 \mid 4n$ if and only if $3 \mid n$. The image is $\{\bar{0}, \bar{4}, \bar{8}\}$.

Exercise S3

Find the positive integers d and e such that

$$8\mathbb{Z} + 12\mathbb{Z} = d\mathbb{Z}, \quad 8\mathbb{Z} \cap 12\mathbb{Z} = e\mathbb{Z}.$$

Solution

$d = 4$ and $e = 24$. The sum is generated by $\gcd(8, 12) = 4$; the intersection consists of the common multiples of 8 and 12.

Exercise S4

In the additive group $\mathbb{Z}/12\mathbb{Z}$, let $H = \langle \bar{4} \rangle$. List the elements of the coset $\bar{1} + H$. Is $\bar{9} + H$ the same coset?

Solution

$H = \{\bar{0}, \bar{4}, \bar{8}\}$, so $\bar{1} + H = \{\bar{1}, \bar{5}, \bar{9}\}$. Yes: $\bar{9} - \bar{1} = \bar{8} \in H$.

Short-answer practice (continued)

Exercise S5

Let $\sigma = (1\ 5\ 2\ 4)(3\ 6) \in S_6$. Find σ^{-1} , the order of σ , and $\text{sgn}(\sigma)$.

Solution

$\sigma^{-1} = (1\ 4\ 2\ 5)(3\ 6)$, and $\text{ord}(\sigma) = \text{lcm}(4, 2) = 4$. Both the 4-cycle and the transposition have sign -1 , so $\text{sgn}(\sigma) = 1$.

Exercise S6

Let G be a group of order 42 and $H \leq G$ a subgroup of order 7. How many left cosets of H are there? Can G contain an element of order 8?

Solution

There are $[G : H] = 42/7 = 6$ left cosets. No element can have order 8, since an element's order divides 42 by Lagrange's theorem, and $8 \nmid 42$.

Exercise S7

Let $\psi : G \rightarrow K$ be a homomorphism, where $|G| = 24$ and $|\ker \psi| = 6$. Find $|\text{im } \psi|$. If $|K| = 8$, is ψ surjective?

Solution

$|\text{im } \psi| = |G|/|\ker \psi| = 24/6 = 4$. If $|K| = 8$, then ψ is not surjective, since its image has only 4 elements.

Exercise S8

Let $H = \{1, (1\ 3)\} \leq S_3$. List the elements of the left coset $(1\ 2)H$ and of the right coset $H(1\ 2)$. Are these cosets equal?

Solution

Composing rightmost first gives

$$(1\ 2)H = \{(1\ 2), (1\ 3\ 2)\}, \quad H(1\ 2) = \{(1\ 2), (1\ 2\ 3)\}.$$

The two cosets are not equal.

True or false

Mark each statement true or false. For review, also try to give a brief reason or a counterexample.

T1. Every cyclic group is abelian.

Solution

True. If $G = \langle a \rangle$, then $a^r a^s = a^{r+s} = a^{s+r} = a^s a^r$.

T2. Every abelian group is cyclic.

Solution

False. The additive group $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ has four elements, but each nonidentity element has order 2.

T3. Every left coset of a subgroup $H \leq G$ is a subgroup of G .

Solution

False. The coset $1 + 2\mathbb{Z}$ in $(\mathbb{Z}, +)$ does not contain the identity 0.

T4. If H is a finite subgroup of G , then every left coset of H has $|H|$ elements.

Solution

True. For any $g \in G$, the map $h \mapsto gh$ is a bijection from H to gH .

T5. If $\varphi : G \rightarrow K$ is a homomorphism and $a \in G$ has finite order, then $\varphi(a)$ has the same order as a .

Solution

False. The trivial homomorphism from a group of order 2 to $\{1\}$ sends its nonidentity element to an element of order 1. The image order need only divide the original order.

True or false (continued)

T6. A group homomorphism with trivial kernel is injective.

Solution

True. If $\varphi(x) = \varphi(y)$, then $\varphi(y^{-1}x) = 1$. A trivial kernel forces $y^{-1}x = 1$, hence $x = y$.

T7. If $a \in G$ has finite order and $g \in G$, then gag^{-1} has the same order as a .

Solution

True. $(gag^{-1})^m = ga^mg^{-1}$, which equals 1 exactly when $a^m = 1$.

T8. The product of two odd permutations is odd.

Solution

False. Sign is multiplicative, and $(-1)(-1) = 1$, so the product is even.

T9. The set of invertible 2×2 real matrices is a group under matrix multiplication.

Solution

True. Products and inverses remain invertible, the identity matrix belongs to the set, and matrix multiplication is associative.

T10. $4\mathbb{Z} \subseteq 6\mathbb{Z}$.

Solution

False. $4 \in 4\mathbb{Z}$, but $4 \notin 6\mathbb{Z}$. In general, $a\mathbb{Z} \subseteq b\mathbb{Z}$ if and only if $b \mid a$.