

Midterm 1 Review Sheet

Algebra 1

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MATH 327

Fall 2026

Date: Friday, October 9, 2026, in class. **Format:** Closed book.

Weight: 25%. **Coverage:** Lectures 1–14; Lecture 14 (October 7) is review/catchup.

TOPICS

1. Groups and subgroups (Lectures 1–2, 6–7).

- Group axioms; abelian groups; identity, inverses, cancellation, and $(ab)^{-1} = b^{-1}a^{-1}$. Distinguish the order of a group from the order of an element.
- Examples: $(\mathbb{Z}, +)$, $\mathbb{Z}/n\mathbb{Z}$ under addition, $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{R})$, and S_n . Check the operation and group axioms; recognize non-examples.
- Subgroup criterion; the finite subgroup criterion; intersections of subgroups. A union of subgroups need not be a subgroup. Generated subgroups $\langle W \rangle$, cyclic subgroups $\langle x \rangle$, and $\mathrm{ord}(x) = |\langle x \rangle|$.
- **Subgroups of $\mathbb{Z}$:** each is $d\mathbb{Z}$ for a unique $d \geq 0$. Review the least-positive-element and division-with-remainder argument. For a, b not both zero, $a\mathbb{Z} + b\mathbb{Z} = \gcd(a, b)\mathbb{Z}$ and $\gcd(a, b) = ma + nb$ for some integers m, n (Bézout). Remember $r \mid s$ if and only if $s\mathbb{Z} \subseteq r\mathbb{Z}$.

2. Permutations and sign (Lectures 2–5, 9).

- Two-line and cycle notation; compose *rightmost first*. Compute products, inverses, powers, and orders. Decompose a permutation into disjoint cycles; disjoint cycles commute.
- Simple transpositions $s_i = (i \ i+1)$ generate S_n . **Length equals the number of inversions:** $\ell(p) = \#\{(i, j) : i < j, p(i) > p(j)\}$. Right multiplication by s_i swaps positions $i, i+1$ and changes this number by $+1$ if $p(i) < p(i+1)$, and by -1 otherwise. Review the bubble-sort proof.
- Parity is well defined; $\mathrm{sgn}(p) = (-1)^{\ell(p)}$ is a homomorphism. Permutation matrices satisfy $[p]e_i = e_{p(i)}$, $[pq] = [p][q]$, and $\det[p] = \mathrm{sgn}(p)$. The alternating group is $A_n = \ker(\mathrm{sgn})$; $|A_n| = n!/2$ for $n \geq 2$.
- Work with S_3 : its six elements, multiplication, generators x, y with $x^3 = y^2 = 1$ and $yx = x^2y$, Cayley graph, and all its subgroups.

3. Homomorphisms and normality (Lectures 7–9).

- Homomorphisms, kernels, images, isomorphisms, and automorphisms. Homomorphisms preserve identity and inverses; kernels and images are subgroups. A homomorphism is injective exactly when its kernel is trivial. The inverse of a bijective homomorphism is a homomorphism.
- Compute kernels and images: reduction $\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$, determinant, and sign. Images of generators must preserve relations, as in the S_3 examples.
- Conjugation $a \mapsto gag^{-1}$ is an automorphism and preserves order. For permutations,

$$\sigma(j_1 \cdots j_r)\sigma^{-1} = (\sigma(j_1) \cdots \sigma(j_r)).$$

- Normality: $gNg^{-1} = N$ for every $g \in G$; every kernel is normal. Test normality in S_3 . Know the definition of a simple group; the classification of finite simple groups was an overview, not a proof developed in these lectures.

4. Cosets, index, and counting (Lectures 10–11).

- Left cosets gH , right cosets Hg , and representatives. G/H denotes the *set of left cosets*; distinguish a coset from its representative. Compute examples in S_3 and $\mathbb{Z}/n\mathbb{Z}$.
- **Coset equality:** $g_1H = g_2H$ if and only if $g_2^{-1}g_1 \in H$. For right cosets, $Hg_1 = Hg_2$ if and only if $g_1g_2^{-1} \in H$. Distinct cosets are disjoint and partition G ; every coset is in bijection with H .
- The map $gH \mapsto Hg^{-1}$ is a well-defined bijection from left to right cosets. The map $gH \mapsto Hg$ need not be well defined. Normality is equivalent to $gH = Hg$ for all g ; it is also equivalent to every left coset being a right coset.
- **Lagrange's theorem:** for finite G , $|G| = [G : H]|H|$. Review the proof by partitioning into equal-size cosets. Thus subgroup orders and element orders divide $|G|$, and a group of prime order is cyclic.
- **Tower law:** if $K \leq H \leq G$ and the indices on the right are finite, $[G : K] = [G : H][H : K]$. Apply it even when the groups themselves are infinite.
- **Counting homomorphisms' fibres:** for $K = \ker \varphi$, $gK \mapsto \varphi(g)$ is a well-defined bijection $G/K \rightarrow \text{im } \varphi$. For finite G , $|G| = |\ker \varphi| |\text{im } \varphi|$.