

MATH 327: Algebra 1

Midterm 1 – Instructor Solutions

Friday, October 9, 2026 Total: 32 points

Short questions

8 points

A. Short answer (2 points each)

A1. In the additive group $\mathbb{Z}/18\mathbb{Z}$, find the order of $\bar{6}$ and list the elements of $\langle \bar{6} \rangle$.

The order is 3, since $3\bar{6} = \bar{18} = \bar{0}$ and neither $\bar{6}$ nor $2\bar{6} = \bar{12}$ is zero. Thus

$$\langle \bar{6} \rangle = \{\bar{0}, \bar{6}, \bar{12}\}.$$

A2. Let $\varphi : \mathbb{Z} \rightarrow \mathbb{Z}/8\mathbb{Z}$ be given by $\varphi(n) = \bar{n}$. Describe $\ker \varphi$ and $\text{im } \varphi$.

$\ker \varphi = 8\mathbb{Z}$ and $\text{im } \varphi = \mathbb{Z}/8\mathbb{Z}$.

B. Multiple choice (1 point each)

B1. Let $H \leq G$ and $a, b \in G$. Which condition is equivalent to $aH = bH$?

(b) $b^{-1}a \in H$. Indeed, $aH = bH$ if and only if $b^{-1}aH = H$, which holds if and only if $b^{-1}a \in H$.

B2. Let $a, b, c \in \mathbb{Z}$ be positive integers. Which statement is not always true?

(d) The claim $a < b \implies a\mathbb{Z} \supseteq b\mathbb{Z}$ is not always true. For example, $2 < 3$, but $3 \in 3\mathbb{Z}$ and $3 \notin 2\mathbb{Z}$. Statements (a) and (c) follow from Bézout's identity, and (b) follows because every subgroup of $\mathbb{Z}$ is cyclic.

C. True or false (1 point each)

C1. Every subgroup of an abelian group is normal.

True. If G is abelian, then $ghg^{-1} = h$ for every $g \in G$ and h in any subgroup $H \leq G$.

C2. The set $M_2(\mathbb{R})$ of all 2×2 real matrices is a group under matrix multiplication.

False. The zero matrix, for example, has no multiplicative inverse.

Question 1

8 points

(a) [2 points] Let $p = (1\ 4\ 2)(3\ 5)$ and $q = (1\ 2\ 3)$ in S_5 . Compute pq as a product of disjoint cycles.

Using rightmost-first composition, pq fixes 1 and sends $2 \mapsto 5 \mapsto 3 \mapsto 4 \mapsto 2$. Hence

$$pq = (2\ 5\ 3\ 4).$$

Marking note: The student exam does not specify a composition convention. With leftmost-first composition, the answer is $(1\ 4\ 3\ 5)$.

(b) [3 points] Let $\sigma = (i_1 \ i_2 \ \cdots \ i_k)$ with k odd. Find its order and parity.

A k -cycle has order k . It is a product of $k - 1$ transpositions, so

$$\text{sgn}(\sigma) = (-1)^{k-1} = 1.$$

Thus σ is an even permutation.

(c) [3 points] Let $a, b \in G$, with a of order 7 and $a^3b = ba^3$. Show that $ab = ba$.

Since $3 \cdot 5 = 15 \equiv 1 \pmod{7}$, we have $a = (a^3)^5$. As $a^3b = ba^3$, it also follows that

$$a^3a^3b = a^3ba^3 = ba^3a^3.$$

Similarly, every power of a^3 commutes with b , so

$$ab = (a^3)^5b = b(a^3)^5 = ba.$$

Question 2

8 points

Let H and K be subgroups of a group G .

(a) [3 points] Prove that $H \cap K$ is a subgroup of G .

The identity belongs to both H and K , so $H \cap K$ is nonempty. If $x, y \in H \cap K$, then $xy^{-1} \in H$ and $xy^{-1} \in K$, because both are subgroups. Therefore $xy^{-1} \in H \cap K$, and the subgroup test gives $H \cap K \leq G$.

(b) [2 points] Give explicit G, H, K for which $H \cup K$ is not a subgroup of G .

Take $G = (\mathbb{Z}, +)$, $H = 2\mathbb{Z}$, and $K = 3\mathbb{Z}$. Both are subgroups of G . Yet $2, 3 \in H \cup K$ while $2 + 3 = 5 \notin H \cup K$. Thus the union is not closed under the group operation.

(c) [3 points] For a homomorphism $\varphi : G \rightarrow G'$, prove that $\ker \varphi$ is normal in G .

The kernel is a subgroup of G . If $g \in G$ and $x \in \ker \varphi$, then

$$\varphi(gxg^{-1}) = \varphi(g)\varphi(x)\varphi(g)^{-1} = \varphi(g)e_{G'}\varphi(g)^{-1} = e_{G'}.$$

Hence $gxg^{-1} \in \ker \varphi$ for all g and x , so $\ker \varphi \trianglelefteq G$.

Question 3

8 points

Let $p < q$ be primes and $|G| = pq$. Show that G has an element of order p without using Cauchy's theorem or a result that directly implies the claim.

(a) [2 points] For $x \in G$ with $x \neq 1$, show that $\text{ord}(x) \in \{p, q, pq\}$.

By Lagrange's theorem, $\text{ord}(x)$ divides pq . Since $x \neq 1$, its order is not 1, leaving p , q , or pq .

(b) [2 points] If some element has order pq , show that G has an element of order p .

If $\text{ord}(x) = pq$, then x^q has order $pq/\gcd(pq, q) = p$.

(c) [4 points] Suppose every non-identity element has order q . Count elements to obtain a contradiction, then conclude.

If Q_1 and Q_2 are distinct subgroups of order q , their intersection is a subgroup of each. Its order divides the prime q ; if it had order q , the two subgroups would be equal. Hence $Q_1 \cap Q_2 = \{1\}$.

Every non-identity element generates a subgroup of order q . Let N be the number of distinct such subgroups. Their non-identity elements are disjoint and cover $G \setminus \{1\}$, so

$$pq - 1 = N(q - 1).$$

But $pq - 1 = p(q - 1) + (p - 1)$, with $0 < p - 1 < q - 1$. Thus $q - 1$ cannot divide $pq - 1$, a contradiction. Therefore some non-identity element has order other than q . By (a), it has order p or pq ; in the latter case, (b) produces an element of order p .